Reliable Semiclassical Computations in QCD

Guido Festuccia

Institute for Advanced Study

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Michael Dine, GF, Lawrence Pack and Weitao Wu

Two questions:

Instantons in QCD

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Lattice QCD

Lattice computations are providing interesting results for QCD in the chiral limit. Look for a test independent of experimental results. Can instantons be useful?

Summary of results

- Study maximally chirality violating green functions in $SU(N)$ QCD with N_f massless flavors.
- Show that for $N_f > N$ the short distance behavior is dominated by a calculable single instanton contribution. The case $N_f = N$ is borderline but a similar result holds for a careful choice of correlators.
- Analyze the consequences of adding small quark masses.
- Provide an estimate of lattice effects to understand when these "observables" can provide a viable calibration of lattice computations.

A two point function

Consider $SU(N)$ QCD with N_f massless flavors

$$\mathcal{O}_1(x) = \prod_{f=1}^{A} \bar{q}_f(x) q_f(x) \qquad \mathcal{O}_2(x) = \prod_{f=A+1}^{N_f} \bar{q}_f(x) q_f(x)$$

Different choices are possible as long as all $q_f, \bar{q}_f$ do appear once.
We will consider the following two point function:

$$\boxed{\Delta(x) = \langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle}$$

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- Maximally chirality violating. Vanishes at any order in perturbation theory.
- Receives a contribution from a single instanton.

Operator Product Expansion

We are interested in the behavior of $\Delta(x)$ for small x. Use the OPE to understand the singularities.

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \approx \sum C_{12}^n(x)\mathcal{O}_n(0)$$

In perturbation theory the lowest dimension operator allowed on the RHS is $\mathcal{O}_{3N_f} = \mathcal{O}_1\mathcal{O}_2$ (and relatives). Its coefficient starts as a constant:

$$C_{3N_f}(x) = c_{3N_f} + O(\alpha_S(x))$$

Non perturbatively at the level of the single instanton the unit operator can appear with coefficient:

$$C^{(0)}(x) \sim \Lambda^{b_0} x^{b_0 - 3N_f}, \qquad b_0 - 3N_f = \frac{11}{3}(N - N_f)$$

Instanton contribution

Look more closely to the instanton contribution:

$$\Delta(x) = C \int d^4x_0 d\rho \frac{(\Lambda\rho)^{\frac{11}{3}N-\frac{2}{3}N_f}\rho^{3N_f-5}}{[(x-x_0)^2+\rho^2]^{3A}[x_0^2+\rho^2]^{3(N_f-A)}}$$

The large ρ behavior is $\Delta \sim \int \frac{d\rho}{\rho}\rho^{\frac{11}{3}(N-N_f)}$

- For $N > N_f$ the ρ integral is IR divergent. The divergence is an (ill-defined) contribution to $\langle \mathcal{O}_{3N_f} \rangle$. The small x behavior is nonsingular.
- For $N < N_f$ the integral is finite. For small x it behaves as $\Delta(x) \sim \Lambda^{b_0} x^{\frac{11}{3}(N-N_f)}$. This is interpreted as a finite contribution to the coefficient of the unit operator in the OPE.
- For $N_f = N$ the ρ integral is logarithmically divergent. The small x behavior is also logarithmic.

- For $N > N_f$ the contribution of $\mathcal{O}_{3N_f}$ in the OPE is the dominant one for small x.

$$\Delta(x) \sim D\,\Lambda^{3N_f}\log(x\Lambda)$$

the coefficient function in the OPE is calculable, but not the matrix element.

- For $N < N_f$ the small x behavior of $\Delta(x)$ is dominated by the identity contribution to the OPE.

$$\Delta(x) \sim C\Lambda^{b_0}x^{\frac{11}{3}(N-N_f)}$$

where the coefficient C can be computed systematically.

- For $N = N_f$ the calculable identity contribution to the OPE can dominate over the contribution of $\mathcal{O}_{3N_f}$ by a power of $\log(x)$ depending on the anomalous dimension of the latter. By choosing $\mathcal{O}_1$ and $\mathcal{O}_2$ this can be arranged for $N = 2, 3$.

The dilute instanton gas

Going beyond the single instanton computation

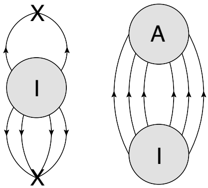
- There are perturbative corrections around the instanton. These are IR finite and modify the the coefficient of the unit operator in the OPE by powers of $\alpha_s(x)$.
- There are dilute gas contributions

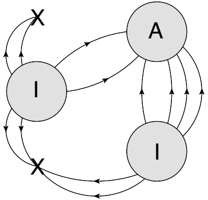
Consider the case of two instantons and one anti-instanton widely separated. We can distinguish two cases

- We can have a paired instanton anti-instanton.
- We can have all instantons connected together and to $\mathcal{O}_1,\ \mathcal{O}_2$.

The dilute instanton gas

- In the first case we get the single instanton result times a factor proportional to the volume V. Corrections of this kind build the exponential e^{-EVT} in the euclidean path integral.
- In the second case the behavior for small x is non singular. All diagrams are infrared divergent. The leading divergence, being x independent, is a correction to $\langle \mathcal{O}_{3N_f} \rangle$.





In general we do not expect dilute gas corrections to give ill defined contributions to the coefficient of the unit operator in the OPE.

$\Delta(x)$ on the lattice

We would like to understand if the small x behavior of $\Delta(x)$ for $N_f > N$ can be extracted from a lattice computation.

- We need to estimate the effect of adding small quark masses.
- It is important to assess when the small x behavior can be discerned over noise.

Use as example the case of $SU(2)$ with three flavors. The instanton contribution evaluated in the $\overline{MS}$ scheme is then

$$\Delta(x) \approx 9 \times 10^3 \mu^{16/3} g^{-8} e^{\frac{-8\pi^2}{g^2(\mu)}} x^{-11/3}$$

we will assume a lattice spacing $a^{-1} = 4GeV$ and $m_q \sim 10 - 20MeV$.

Quark masses

For nonzero quark masses generically there is a perturbative contribution:

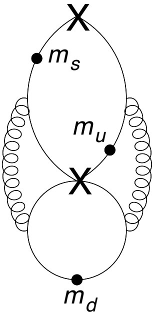
$$\Delta = \frac{m_u m_d m_s}{a^6}$$

which depends strongly on the cutoff $\frac{1}{a}$ and swamps the instanton contribution.

With parity odd operators e.g.

$$\mathcal{O}_1 = \bar{d}\gamma_5 d\bar{u}s, \quad \mathcal{O}_2 = \bar{s}\gamma_5 u$$

the first nonzero perturbative contribution is at four loops and sufficiently small:



$$\Delta = c\left(\frac{\alpha_s}{4\pi}\right)^2 \frac{m_u m_d m_s}{x^6}$$

Is there hope to extract $\Delta(x)$ for small x over noise?

Very roughly we can compare to

$$\Delta' = S_F^3 \sim \left(\frac{1}{\pi^2}\right)^3 |x|^{-9}$$

this is comparable to $\Delta(x)$ for $x^{-1} \sim 1.5 GeV$

Two point functions like the ones we studied can be constructed for matter in different representations of the gauge group.
For SU(2) with two Dirac adjoint fermions we can consider $\mathcal{O}_1$, $\mathcal{O}_2$ with 8 fermions each.

$$b_0 = 2 \times \frac{11}{3} - 8 \times \frac{2}{3} = 2 \Rightarrow \Delta(x) \sim \Lambda^{b_0} x^{-22}$$

For SU(3) with one Dirac fermion in $\Box\!\Box$ use $\mathcal{O}_1$, $\mathcal{O}_2$ with 5 fermions:

$$b_0 = 3 \times \frac{11}{3} - 5 \times \frac{2}{3} = \frac{23}{3} \Rightarrow \Delta(x) \sim \Lambda^{b_0} x^{-\frac{22}{3}}$$

In this cases also the unit operator dominates at small separation.

Conclusions

- We presented observables which do not receive any perturbative contribution in QCD like-theories in the massless limit.
- For $N_f \geq N$ they can be computed semiclassically in a systematic way.
- We investigated the possibility of using these observables as tests of lattice computations in the chiral limit. The results allow cautious optimism.
- Further study of lattice effects could help in refining the observables.

Thank You !