GMOR relation in IR-conformal theories

Agostino Patella

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Intro

- QCD in the chiral limit: χS is spontaneously broken. When a small mass is given to the fermions, the Gell-Mann-Oaks-Renner showed that

$$m\Sigma \simeq F_{PS}^2 M_{PS}^2$$

- Consider a theory in the conformal window, and give a small mass to fermions. How is the GMOR relation modified in this case?

- A generalized version of the GMOR can be obtained using RG-analysis and the PCAC (Partially Conserved Axial Current) relations.

GMOR relation

- χSB $\qquad \Sigma \neq 0 \qquad M_{PS} \to 0 \qquad F_{PS} > 0$

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- IR-conformality $1 < \gamma_* < 2$ $\qquad \Sigma \sim m^{\frac{3-\gamma_*}{1+\gamma_*}} \qquad M_{PS}, F_{PS} \sim m^{\frac{1}{1+\gamma_*}}$

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$$\lim_{m \to 0} \frac{m\Sigma - m\Sigma'(m=0)}{M_{PS}^2 F_{PS}^2} = \text{finite}$$

Partially Conserved Axial Current

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- On shell $\qquad \partial_\mu A_\mu(y) = 2mP(x)$

Källén-Lehmann spectral representation for the PS correlator:

$$C_{PS}(x,m) = \frac{F_{PS}^2 M_{PS}^4}{2m^2}\Delta(x, M_{PS}^2) + \int_{S_{PS}}^{\infty} R_{PS}(s,m)\Delta(x,s)ds$$

$$\langle 0|A_\mu(0)|\pi, p_\mu\rangle = F_{PS}p_\mu$$

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- Off shell $\qquad P^\dagger(x)\partial_\mu A_\mu(y) - 2mP^\dagger(x)P(y) = (\bar{u}u + \bar{d}d)(x)\delta^4(x-y)$

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The GMOR relation follows from the integrated PCAC relation.

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DIVERGENT!

The GMOR relation follows from the integrated PCAC relation.

Analysis of divergences

$$m \int \langle P^\dagger(x) P(0) \rangle d^4x = \Sigma$$

- PS susceptibility. At small x: $\langle P^\dagger(x) P(0) \rangle \sim |x|^{-6}$
- Chiral condensate. Let $i\omega_a[A]$ be the generic eigenvalue of $\not{D} = \gamma_\mu D_\mu[A]$

$$\rho(\omega, m) = \lim_{L \to \infty} \frac{1}{L^4} \sum_a \langle \delta(\omega - \omega_a) \rangle$$

$$\Sigma = 2m \int_0^\infty \frac{\rho(\omega, m)}{m^2 + \omega^2} d\omega$$

At large ω: $\rho(\omega, m) \sim \omega^3$

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One can easily regularize the chiral condensate by introducing a cutoff in the integral. Is it possible to regularize the PS susceptibility in such a way that the identity still holds?

Master equation

I want to find a regularized version of the following relation:

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and separate the leading IR behaviour from the UV divergences.

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$$m \int \langle \bar{d}\gamma_5 u(x) \bar{u}\gamma_5 d(0) \rangle d^4x = \langle \bar{u}u \rangle$$

Integrate out fermions

$$m \int \langle \mathrm{tr}\ \gamma_5 \frac{1}{\not{D} + m}(x,0)\gamma_5 \frac{1}{\not{D} + m}(0,x) \rangle d^4x = \langle \mathrm{tr}\ \frac{1}{\not{D} + m}(0,0) \rangle$$

Cut the eigenvalues of the Dirac operator (only in observables)

$$\not{D} \rightarrow \not{D}(\omega < \omega_\infty)$$

$$m \int C_{PS}^{\mathcal{R}}(x, m, \omega_\infty) d^4x = 2m \int_0^{\omega_\infty} \frac{\rho(\omega, m)}{m^2 + \omega^2} d\omega \equiv \Sigma^{\mathcal{R}}(m, \omega_\infty)$$

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Split the integrals

$$m \int_0^T + \int_T^\infty dt \int d^3\mathbf{x}\, C^{\mathcal{R}}_{PS}(x, m, \omega_\infty) = 2m \int_0^{\bar{\omega}} + \int_{\bar{\omega}}^{\omega_\infty} \frac{\rho(\omega, m)}{m^2 + \omega^2} d\omega$$

Move all the UV divergences to the rhs

$$2m \int_T^\infty dt \int d^3\mathbf{x}\, C^{\mathcal{R}}_{PS}(x, m, \omega_\infty) - \Sigma^{\mathcal{R}}(m, \bar{\omega}) =$$
$$= 2m \int_{\bar{\omega}}^{\omega_\infty} \frac{\rho(\omega, m)}{m^2 + \omega^2} d\omega - m \int_0^T dt \int d^3\mathbf{x}\, C^{\mathcal{R}}_{PS}(x, m, \omega_\infty)$$

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$$\int_T^\infty dt \int d^3\mathbf{x}\, C_{PS}(x,m) - \frac{\Sigma^{\mathcal{R}}(m,\bar{\omega})}{2m} =$$
$$= \lim_{\omega_\infty \to \infty} \left[\int_{\bar{\omega}}^{\omega_\infty} \frac{\rho(\omega,m)}{m^2+\omega^2} d\omega - \int_0^T dt \int d^3\mathbf{x}\, C_{PS}^{\mathcal{R}}(x,m,\omega_\infty) \right]$$

χSB

Which pieces are divergent in the chiral limit?

GREEN: Finite

BLUE: Subleading divergence

RED: Leading divergence ($\sim m^{-1}$)

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$$\lim_{m\to 0} \frac{F_{PS}^2 M_{PS}^2}{m} + \lim_{m\to 0} 2m \int_{S_{PS}}^\infty R_{PS}(s,m) e^{-T\sqrt{s}} \frac{ds}{s} = \Sigma_0$$

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$$\lim_{m\to 0} \frac{m\Sigma^{\mathcal{R}}(m,\bar{\omega})}{F_{PS}^2 M_{PS}^2} = 1 + \lim_{m\to 0} \frac{2m^2}{F_{PS}^2 M_{PS}^2} \int_{S_{PS}}^\infty R_{PS}(s,m) e^{-T\sqrt{s}} \frac{ds}{s}$$

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The PS susceptibility is IR finite!!! One can take derivatives with rispect to the mass.

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Summary

- χSB

$$\lim_{m\to 0} \frac{m\Sigma(\bar{\omega})}{M_{PS}^2 F_{PS}^2} = 1$$

- IR-conformality $1 < \gamma_* < 2$

$$\lim_{m\to 0} \frac{m\Sigma(\bar{\omega})}{F_{PS}^2 M_{PS}^2} = 1 + \lim_{m\to 0} \frac{2m^2}{F_{PS}^2 M_{PS}^2} \int_{S_{PS}}^{\infty} R_{PS}(s,m) e^{-T\sqrt{s}} \frac{ds}{s} > 1$$

- IR-conformality $0 < \gamma_* < 1$

$$\lim_{m\to 0} \frac{m\Sigma(\bar{\omega})}{F_{PS}^2 M_{PS}^2} = \infty$$

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Lattice implementation with Wilson fermions

How to measure $\Sigma(\bar{\omega})$ on the lattice? If D_W is the Dirac-Wilson operator...

$$\hat{\Sigma}(\hat{g}, \hat{m}, \hat{\omega}) = \frac{\hat{m}}{\hat{V}} \langle \mathrm{tr}\, \frac{\hat{\mathbb{P}}(\hat{m}^2 + \hat{\omega}^2)}{D_W^\dagger D_W} \rangle$$

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How do you keep the renormalised cutoff ω fixed while taking the continuum limit? One can use a RG-invariant quantity, like the mode number:

$$\hat{\zeta}(\hat{g}, \hat{m}, \hat{\omega}) = \lim_{\hat{V} \to \infty} \frac{1}{\hat{V}} \langle \mathrm{tr}\, \hat{\mathbb{P}}(\hat{m}^2 + \hat{\omega}^2) \rangle \ .$$

which is the lattice version of the following quantity:

$$\zeta(m, \omega) = \lim_{V \to \infty} \frac{1}{V} \int_0^\omega \rho(\tilde{\omega}, m) d\tilde{\omega}$$

Lattice implementation with Wilson fermions

Recipe:

1. Choose the physical values for the Sommer radius r_0, the PS mass M_{PS} and the mode number ζ.
2. Choose a value for $\hat{g}$, measure the lattice version of the Sommer radius $\hat{r}_0$, and define the lattice spacing $a = r_0/\hat{r}_0$.
3. Measure the lattice version of the PS mass for several bare masses, and tune the bare mass in order to get $\hat{M}_{PS} = aM_{PS}$.
4. Measure the lattice version of the mode number for several bare cutoffs, and tune the bare cutoff in order to get $\hat{\zeta} = a^4\zeta$.
5. Measure $\hat{\Sigma}(\hat{g}, \hat{m}, \hat{\omega})$.
6. Restart from point 2 with a smaller $\hat{g}$.

$$\frac{m\Sigma(m,\omega)}{F_{PS}^2 M_{PS}^2} = \lim_{\hat{g}\to 0} \frac{\hat{m}\hat{\Sigma}(\hat{g},\hat{m},\hat{\omega})}{\hat{F}_{PS}^2 \hat{M}_{PS}^2}$$