Index Theorem and Random Matrix Theory for Improved Staggered Quarks

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We study various improved staggered quark Dirac operators on quenched gluon backgrounds in lattice QCD generated using a Symanzik-improved gluon action. We find a clear separation of the spectrum of eigenvalues into would-be zero modes and others. The number of would-be zero modes depends on the topological charge as expected from the Index Theorem, and their chirality expectation value is large. The remaining modes have low chirality and show clear signs of clustering into quartets and approaching the random matrix theory predictions for all topological charge sectors. We conclude that improvement of the fermionic and gauge actions moves the staggered quarks closer to the continuum limit where they respond correctly to QCD topology.

1. INTRODUCTION

Topology plays a key role in our understanding of some important features of QCD, such as the axial anomaly, or the η' mass.

In the continuum, the Index Theorem relates the topological charge of a smooth gauge field with the number of chiral zero modes of the corresponding Dirac operator.

For QCD in the ϵ regime, there are detailed predictions of the distribution of the low-lying eigenvalues of the Dirac operator, in each sector of fixed topological charge.

A correct discretization of QCD must reproduce these features, at least in the continuum limit. Furthermore, in order to use such discretizations in practice to tackle calculations related to topology, we would like to see such topological properties manifesting themselves at values of the parameters at which we can realistically do simulations.

It has been widely held that the staggered discretization of QCD is insensitive to topology. Previous studies did not find, in the raw, non-smoothed gauge field configurations, signs of an Index Theorem, and the predictions of RMT were not reproduced; the calculations in all sectors of fixed topological charge gave the same results, in agreement with the theoretical predictions for the sector of zero topological charge.

There are large-scale simulations using staggered fermions today [1,2], and more are planned for the near future. It is therefore important to understand to what extent staggered quarks show the correct topological properties.

2. STAGGERED DIRAC OPERATORS

All the Dirac operators we study have the general form:

$$S = \sum_{x,y} \bar{\chi}(x) \not{D}(x,y) \chi(y) \tag{1}$$

with $\not{D}$ a gauge-invariant linear operator. The first example is the one-link staggered Dirac operator (also called naive, unimproved staggered or Kogut-Susskind in the literature):

$$\not{D}(x,y) = \tfrac{1}{2u_0} \textstyle\sum_\mu \alpha_\mu(x)(U_\mu(x)\delta_{x+\mu,y} - H.c.) \tag{2}$$

$$\alpha_\mu(x) = (-1)^{\sum_{\nu>\mu} x_\nu} \tag{3}$$

This Dirac operator has some simple properties, which are shared with all the improved operators we will introduce later. First, it is antihermitian, $\not{D}^\dagger = -\not{D}$. It also obeys a remnant of the continuum γ_5 anticommutation relation:

$$\{\not{D}, \epsilon\} = 0, \quad \text{with} \quad \epsilon(x) = (-1)^{\sum_\mu x_\mu} \tag{4}$$

*The results presented here have been obtained in collaboration with A. Hart and C. Davies.

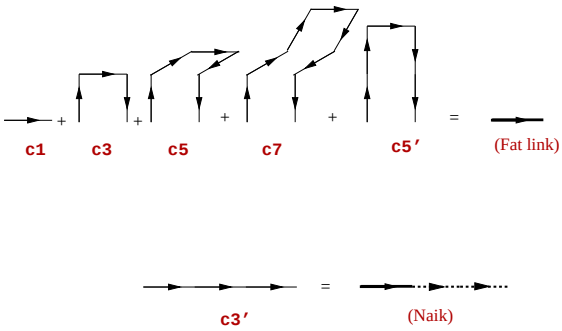

Figure 1. paths entering the ASQ(TAD) operator

As a consequence of these two properties, its spectrum is purely imaginary and eigenvalues come in complex-conjugate pairs,

$$sp(\not{D}) = \{\pm i\lambda, \lambda \in \Re\} \tag{5}$$

This operator corresponds (in four dimensions) to 4 "tastes" of fermions. There are unphysical taste-changing interactions, involving at leading order the exchange of a gluon of momentum $q \approx \pi/a$. Such interactions are perturbative for typical values of the lattice spacing, and can be corrected systematically à la Symanzik. This can be accomplished by smearing the gauge field to remove the coupling between quarks and gluons with momentum π/a [3–7]. Including appropriate paths up to length seven leads to the so-called FAT7 operator. By adding two more terms (the five-link Lepage term and the three-link Naik term), we obtain an operator improved to order a^2 at tree level, called ASQ. If we also add tadpole improvement we get the ASQTAD operator.

Another improved staggered Dirac operator, motivated by perfect action ideas, is the HYP operator, which involves three levels of (restricted) APE smearing with projection onto $SU(3)$ at each level. The restrictions are such that each fat link includes contributions only from thin links belonging to hypercubes attached to the original link. [8].

The final improved staggered operator we will consider here is the so called HISQ (Highly Improved Staggered Quarks) operator, which involves two levels of smearing: first a FAT7 smearing on the original links, followed by a projection onto $SU(3)$, then ASQ on these fattened links [9,10].

Both HYP and HISQ show much smaller taste-changing effects than ASQTAD.

Table 1
Parameters of the improved gauge ensembles

Volume	12^4	16^4	12^4
a (fm)	0.093	0.093	0.125
Length (fm)	1.12	1.49	1.50

3. DETAILS OF THE SIMULATION

Most of our results come from gauge configurations generated using a gauge action which is Symanzik-improved at tree level, including tadpole improvement. We have three different ensembles of about 1000 configurations each, whose parameters are shown in table 1. We will refer to the configurations by their volume, and whether they correspond to the fine, $a = 0.093$ fm ensemble or to the coarse, 0.125 fm one. The topological charge Q is determined by cooling the fields and then using a highly accurate discretization of the continuum expression. We do the cooling with two different actions to check for consistency. The value of Q for lattice gauge fields is not, in general, unambiguously defined, and therefore the values obtained with the two methods do not always coincide. This happens in about 10% of the configurations.

We have also used a few Wilson (unimproved) gauge configurations, with a lattice spacing of $a = 0.093$, and a volume of $16^3 \times 32$.

4. SPECTRUM AND INDEX THEOREM

4.1. Continuum

The eigenmodes of the anti-hermitian, gauge-covariant, massless continuum Dirac operator are given by

$$\not{D} f_s = i\lambda_s f_s \,, \qquad \lambda_s \in R \,. \tag{6}$$

where we use orthonormalised eigenvectors, $f_s^\dagger f_t = \delta_{s,t}$. As $\{\not{D}, \gamma_5\} = 0$, the spectrum is symmetric about zero: if $\lambda_s \neq 0$, then $\gamma_5 f_s$ is also an eigenvector with eigenvalue $-i\lambda_s$, and chirality $\chi_s \equiv f_s^\dagger \gamma_5 f_s = 0$. The zero modes, $\lambda_s = 0$, can be chosen with definite chirality: $\chi_s = \pm 1$. In general there are $n^\pm$ such modes, whose relative number is fixed by the (gluonic) topological charge

$$Q = \frac{1}{32\pi^2} \int d^4x \; \epsilon_{\mu\nu\sigma\tau} \; \mathrm{Tr}\, F_{\mu\nu}(x) F_{\sigma\tau}(x) \tag{7}$$

via the Atiyah–Singer Index Theorem [11,12]

$$Q = m \; \mathrm{Tr} \, \frac{\gamma_5}{\not{D} + m} = n^+ - n^- \; , \tag{8}$$

where m is the quark mass.

4.2. Chiral lattice discretizations

On the lattice, Dirac operators that satisfy the Ginsparg-Wilson relation

$$\{\gamma_5, \not{D}\} = \overline{a} \not{D} \gamma_5 \not{D} \tag{9}$$

can have exact, chiral zero modes, which then may be used to define a topological charge [13]:

$$Q \;\; = a^4 \textstyle\sum_x q(x) = n^+ - n^- \tag{10}$$

$$q(x) \;\; = -\tfrac{1}{2}\, \overline{a} \;\; \mathrm{Tr}\, \{\gamma_5 \not{D}(x,x)\} \tag{11}$$

where $q(x)$ is a local, gauge-invariant function of the gauge fields.

For the fixed-point Dirac operator, furthermore, there is a genuine Index Theorem at finite cutoff [13],

$$Q^{FP} = n^+ - n^- \tag{12}$$

where Q_{FP} is a purely gluonic operator with the characteristics of a proper topological charge.

4.3. Staggered discretization

The staggered Dirac operator has no exact zero modes, and therefore we cannot expect to have an exact Index Theorem. However, close to the continuum limit, we should see an approximate version of the continuum behaviour: the first few eigenmodes with high chirality, in the number required by the continuum Index Theorem, and the rest of the eigenmodes with small chirality.

This is known to happen with sufficiently smooth gauge fields , obtained, for example, by repeated smearing [14,15], or by a lattice discretization of continuum instantons [16].

However, we are interested here in the properties of the raw, non-smoothed configurations, where such behaviour was not found in previous studies with unimproved staggered quarks [17–19]. We want to understand to what degree the continuum features are already present in gauge fields from ensembles generated with parameters used in typical present-day simulations.

To test this we compute the topological charge and the chirality of the first few eigenmodes for the gauge fields in our ensembles. The staggered version of the continuum γ_5 operator relevant for this calculation must be a taste-singlet operator, in order for it to couple to the vacuum. We use a gauge-invariant, point-split four link operator [20].

Taking into account that the staggered discretization describes 4 tastes, we expect to have a quadruple near-degeneracy in the spectrum. Furthermore, if an approximate Index Theorem applies, we expect to see 4 n^+ approximate zero modes with chirality near 1, and 4 n^- approximate zero modes with chirality near -1, with n^+, n^- such that

$$Q = \frac{1}{4}(n^+ - n^-) \tag{13}$$

We show in fig. 2 and 3, for several staggered Dirac operators, the absolute value of the first low-lying eigenvalues, as well as the corresponding chirality, for typical configurations with topological charge $Q = 2$. Fig. 2 corresponds to a configuration generated with the Wilson gauge action, whereas fig. 3 is for the improved gauge action. Due to the exact symmetry (5), we plot only half of the spectrum (in particular, there are an equal number of near-zero modes in the other half, and therefore only $2Q$ of them are seen in the figures.)

We can see a strong difference between the Wilson and the improved gauge configurations. For the one-link operator neither of them show much of the continuum-like behaviour. As we improve the operator, the agreement for the Wilson glue

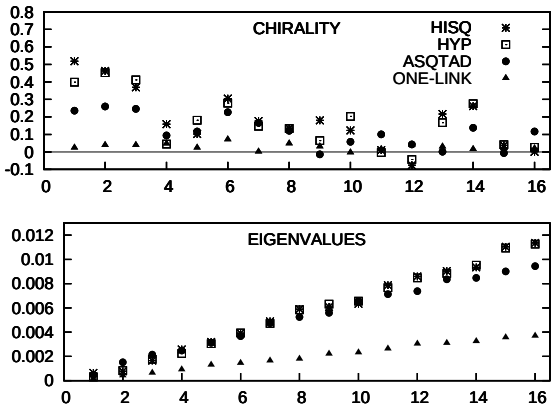

Figure 2. The absolute value of a typical low-lying eigenmode (half) spectrum for a $16^3 \times 32$ Wilson gauge configuration of $Q = 2$, for various staggered fermion formulations. The bottom panel gives the absolute value of the eigenvalue, λ_s, ordered according to increasing size. The x axis is then simply eigenvalue number. The top panel is the chirality of the modes.

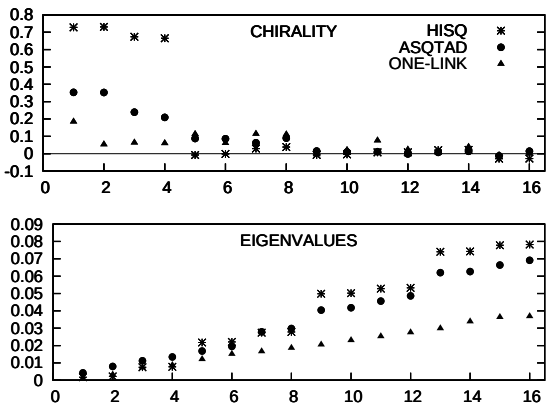

Figure 3. as in fig. 2, for a fine 16^4 improved gauge configuration with $Q = 2$. The HYP action gives results very similar to HISQ and is not plotted here for clarity.

is at best qualitative, showing an increased chirality for the low eigenmodes, and a hint of the expected degeneracy in the spectrum. For the improved gauge glue, however, we see the non-zero eigenvalues clearly grouping in quadruplets, and a sharp separation between chiral and non-chiral modes. The Index Theorem is well approximated for the HISQ operator, with the expected number of near-chiral modes.

To give an idea of how generic this behaviour is, we show in fig. 4 a scatter plot of the absolute value of the chirality vs the absolute value of the eigenvalues for a number of improved gauge configurations. As we improve the Dirac operator, a gap develops between the high-chirality, near-zero modes, and the low-chirality, non-zero modes. The chirality of the near-zero modes is remarkably constant over the different configurations, with a value of around 0.7. The separation is not strict, however, and even for the HISQ operator there are configurations with low modes of intermediate chirality. If we choose some arbitrary threshold in the chirality to count zero modes, let's say over .65 in absolute value, we can then use the Index Theorem to assign a fermionic topological charge to the gauge field, $Q_F = n^+ - n^-$. This charge coincides with the one measured via cooling in about 90% of the cases, and therefore the ambiguity in this definition is about the same as there is between the two cooling methods for this value of the lattice spacing.

In figure 5 we show the effect on the spectrum of changing the volume (keeping the lattice spacing constant) while keeping the Dirac operator fixed, compared with changing the operator at constant volume. As before, we need to use the improved operator for see any degeneracy. However the lattice volume has a strong effect too, and the degeneracy is most clear at the smaller volume. If we were to increase the volume further, eventually the degeneracy would not be evident any more (the spectrum will become dense in the infinite volume limit). However, even when not obvious, it may still be present in some form, as will be clear in our results for the ratios of eigenvalues.

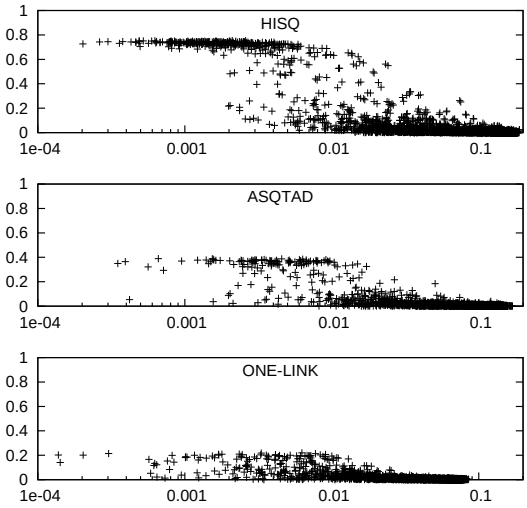

Figure 4. A scatter plot for different staggered quark formulations, with absolute value of the chirality on the y axis and eigenvalue λ_s on the x axis. The lowest 50 eigenvalues for 147 configurations are plotted.

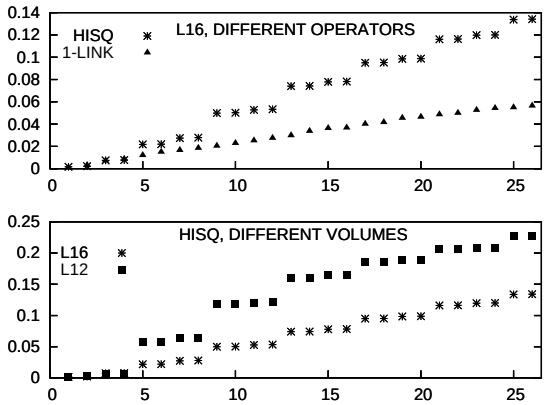

Figure 5. Low lying (half) spectrum. The top panel is a comparison of the one-link and the HISQ spectrum for a configuration in the fine 16^4 ensemble. The bottom panel shows the HISQ spectrum for the same configuration above, and for a configuration at the same lattice spacing and with 12^4 volume.

5. RANDOM MATRIX THEORY PREDICTIONS

Based on [21], it has been suggested that, in the ϵ regime of QCD (volume very large, but much smaller than the pion length scale), and in each sector of fixed topological charge Q, the non-zero low-lying eigenmodes, appropriately scaled, take values from a universal distribution, which only depends on Q. [22]. The universality class is determined by the chiral symmetries of QCD [2]. The distributions can be derived from any theory in the correct universality class, such as ensembles of random matrices [23,14] (for a review of other theories, see [15]).

These predictions have been succesfully tested in lattice QCD with Ginsparg-Wilson fermions [24–29].

On the other hand, previous studies with unimproved staggered fermions (on much coarse lattices and with Wilson action gauge fields) had shown a very different behaviour, with the eigenvalues for any Q behaving according to the theoretical prediction for the $Q = 0$ sector [3] [30–34].

In order to test the universality predictions, first we subtract from the spectrum the $4Q$ lowest eigenvalues ($2Q$ on either side of zero), which should converge to zero modes in the continuum limit, according to the Index Theorem. We then group the remaining eigenvalues, ordered by size, into sets of four, corresponding to the four-fold degeneracy of the spectrum in the continuum limit. We then average the values in each quartet, and denote the resulting averages $\Lambda_1, \Lambda_2, \ldots$. In fig. 6 we plot the ratios (denoted by "s/t") $\langle \Lambda_s \rangle_Q / \langle \Lambda_t \rangle_Q$ where the expectation values $\langle \cdot \rangle_Q$ are over the sectors with gluonic topological charges $\pm Q$ only. The universal predictions for this ratios (which are independent of any scale) are also shown on the figure.

The first thing to notice is the clear dependence on Q, in stark contrast with previous results. The ratios are systematically below the

[2] And also the number of dimensions, the gauge group and the representation in which the fermions lie.

[3] It should be noted that the method we follow here, of grouping the eigenvalues into quartets, was not followed because this feature of the spectrum was not evident.

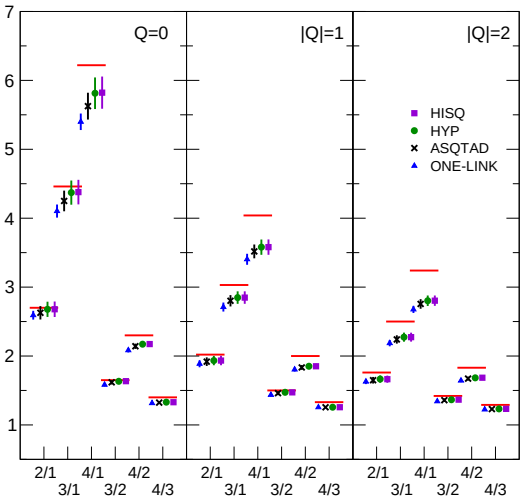

Figure 6. The ratios of expectation values of small eigenvalues (see text for notation) compared with the predictions based on a universal distribution (horizontal lines) for topological charge sectors 0, 1 and 2.

theoretical predictions, especially the ones involving higher eigenvalues. This would be consistent with finite volume effects as in [29]. There is also a small but systematic difference between the one-link and the improved actions, with the improved results showing a better agreement with the theoretical values. As in [29] we find no significant changes on the coarse lattice at the same V.

An important point to make here is that it is necessary to group the eigenvalues as explained above to get sensible results. If one ignores the near zero modes, or does not group in quartets, ratios which are close to one or very large will result. This is strong evidence that the four tastes are showing up in the spectrum, even where it is not directly evident in the spectrum itself.

As we discussed before, there is a (small) ambiguity in the determination of the topological charge, so that the two cooling methods sometimes give a different answer for Q. One can then either choose any one of the two methods for defining Q, or use in the analysis only the configurations for which both methods agree. We have checked that it makes no difference to within our statistical accuracy.

6. CONCLUSIONS AND OUTLOOK

Improved staggered fermions are not blind to the topology, but in fact reproduce well the predictions of the Index Theorem, and the universality of ratios of eigenvalues as a function of topological sector. This means we can have confidence in using them to attack the questions arising from the axial anomaly in QCD.

We also remark that the fact that the 4-fold taste degeneracy of staggered quarks is becoming clear in the spectrum is encouraging for the programme of establishing the effect of taking the fourth root of the staggered determinant to represent one flavour of staggered sea quarks. This programme requires an analysis in the taste basis and progress towards this is now possible.

More extensive studies of finite volume and lattice spacing effects and analysis of the eigenvectors are underway and will be reported elsewhere.

Finally we would like to point out two talks on related topics in this conference [35,36], showing similarly encouraging aspects of the staggered eigenvalue spectrum.

7. ACKNOWLEDGEMENTS

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