Exact supersymmetry on the lattice: the Wess-Zumino model

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Exact supersymmetry on the lattice

The lattice action is not unique. Improve the action in order to approach the continuum limit faster and/or have less symmetry breaking.

Improving lattice supersymmetry seems to be a difficult task for gauge theories because on the lattice the gauge field and the fermions are treated in a very different way.

It is possible to obtain perfect supersymmetry respect to the supersymmetric transformations. Some examples:

Golterman & Petcher, Nucl.Phys.B319:307-341,1989
Bietenholz, Mod.Phys.Lett.A14:51-62,1999
Catterall & Karamov, Phys.Rev.D65:094501,2002,
Phys.Rev.D68:014503,2003
Fujikawa & Ishibashi, Phys.Lett.B528:295-300,2002
Fujikawa, Nucl.Phys.B636:80-98,2002
Beccaria, Campostrini & A. F., Phys.Rev.D69:095010,2004 and
hep-lat/0405016 (Phys.Rev.D. to appear)
Bonini & A. F., hep-lat/0402034, D'adda et al., hep-lat/0406029

Four dimensional lattice Wess-Zumino model with GW fermions

Our starting point is the paper by Fujikawa (2002)

We show that it is actually possible to formulate the theory in such a way that the full action is invariant under a lattice superymmetry transformation at fixed lattice spacing.

The action and the transformation are written in terms of the Ginsparg-Wilson operator and reduce to their continuum expression in the limit $a \to 0$.

The lattice supersymmetry transformation is non-linear in the scalar fields and depends on the parameters m and g entering in the superpotential.

We also show that the lattice supersymmetry transformation close the algebra, a necessary ingredient to guarantee the request of supersymmetry.

The Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

implies a continuum symmetry of the fermion action which may be regarded as a lattice form of the chiral symmetry (Lüscher 1998).

The fermion lagrangian with a Yukawa interaction

$$\mathcal{L} = \bar{\psi} D\psi + g\bar{\psi}(P_+\phi\hat{P}_+ + P_-\phi^\dagger\hat{P}_-)\psi\,,$$

where

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)\,, \qquad \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5)$$

are the lattice chiral projection operators and $\hat{\gamma}_5 = \gamma_5(1 - aD)$, is invariant under the lattice chiral transformation

$$\delta\psi = i\varepsilon\hat{\gamma}_5\psi\,, \qquad \delta\bar{\psi} = i\bar{\psi}\gamma_5\varepsilon\,, \qquad \delta\phi = -2i\varepsilon\phi\,.$$

By writing ψ in terms of two Majorana fermions

$$\psi = \chi + i\eta \,,$$

it can be seen that the interaction term couples the two Majorana fermions and therefore there is a conflict between lattice chiral symmetry and the Majorana condition (Fujikawa 2002).

This is due to the fact that the projection operators $\hat{P}_{\pm}$ depend on D. By making the following field redefinition

$$\psi' = (1 - \frac{a}{2}D)\psi \,, \qquad \bar{\psi}' = \bar{\psi} \,,$$

the Yukawa interaction becomes

$$g\bar{\psi}'(P_{+}\phi P_{+} + P_{-}\phi^{\dagger}P_{-})\psi'$$

and the two Majorana components of ψ' decouple.

Taking advantage of this property, one can define the four dimensional Wess-Zumino on the lattice with Majorana fermions.

We start with a lagrangian defined in terms of the Ginsparg-Wilson fermions on the $d = 4$ euclidean lattice. A simple solution vas given by Neuberger (1998)

$$D = \frac{1}{a}\big(1 - \frac{X}{\sqrt{X^\dagger X}}\big)\,, \qquad\qquad X = 1 - aD_w\,,$$

where

$$D_w = \frac{1}{2}\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - \frac{a}{2}\nabla^\star_\mu\nabla_\mu$$

and

$$\nabla_\mu\phi(x) = \frac{1}{a}(\phi(x + a\hat{\mu}) - \phi(x))\,, \qquad \nabla^\star_\mu\phi(x) = \frac{1}{a}(\phi(x) - \phi(x - a\hat{\mu}))$$

It is convenient to write

$$D = D_1 + D_2$$

where

$$D_1 = \frac{1}{a}\left(1 - \frac{1 + \frac{a^2}{2}\nabla^\star_\mu \nabla_\mu}{\sqrt{X^\dagger X}}\right), \qquad D_2 = \frac{1}{2}\gamma_\mu \frac{\nabla^\star_\mu + \nabla_\mu}{\sqrt{X^\dagger X}} \equiv \gamma_\mu D_{2\mu}\,.$$

In terms of D_1 and D_2 the Ginsparg-Wilson relation becomes

$$D_1^2 - D_2^2 = \frac{2}{a} D_1\,.$$

The action of the 4-dimensional Wess-Zumino model on the lattice

$$S_{WZ} = \sum_x \{ \frac{1}{2}\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2\chi - \frac{2}{a}\phi^\dagger D_1 \phi + F^\dagger(1-\frac{a}{2}D_1)^{-1}F + \frac{1}{2}m\bar{\chi}\chi$$
$$+m(F\phi + (F\phi)^\dagger) + g\bar{\chi}(P_+\phi P_+ + P_-\phi^\dagger P_-)\chi + g(F\phi^2 + (F\phi^2)^\dagger)\} ,$$

where ϕ and F are scalar fields and χ is a Majorana fermion which satisfies the Majorana condition

$$\bar{\chi} = \chi^T C$$

and C is the charge conjugation matrix which satisfies

$$C^T = -C , \qquad CC^\dagger = 1 .$$

Moreover, our conventions are

$$C\gamma_\mu C^{-1} = -(\gamma_\mu)^T$$
$$C\gamma_5 C^{-1} = (\gamma_5)^T .$$

In the continuum limit reduces to the continuum Wess-Zumino action

$$S = \int \{\frac{1}{2}\bar{\chi}(\partial\!\!\!/ + m)\chi + \phi^\dagger\partial^2\phi + F^\dagger F + m(F\phi + (F\phi)^\dagger) + g\bar{\chi}(P_+\phi P_+ + P_-\phi^\dagger P_-)\chi + g(F\phi^2 + (F\phi^2)^\dagger)\}.$$

The supersymmetric transformation

If one defines the real components by

$$\phi \to \frac{1}{\sqrt{2}}(A + iB)\,, \quad F \to \frac{1}{\sqrt{2}}(F - iG)$$

the WZ model $S_{WZ} = S_0 + S_{int}$

$$S_0 = \sum_x \{\frac{1}{2}\bar{\chi}(1 - \frac{a}{2}D_1)^{-1}D_2\chi - \frac{1}{a}(AD_1A + BD_1B) + \frac{1}{2}F(1 - \frac{a}{2}D_1)^{-1}F + \frac{1}{2}G(1 - \frac{a}{2}D_1)^{-1}G\}\,,$$

$$S_{int} = \sum_x \{\frac{1}{2}m\bar{\chi}\chi + m(FA + GB) + \frac{1}{\sqrt{2}}g\bar{\chi}(A + i\gamma_5 B)\chi + \frac{1}{\sqrt{2}}g[F(A^2 - B^2) + 2G(AB)]\}\,.$$

S_0 (free part) is invariant under the lattice supersymmetry transformation (Fujikawa 2002)

$$\begin{aligned}
\delta A &= \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
\delta B &= -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
\delta\chi &= -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon \\
\delta F &= \bar{\varepsilon}D_2\chi \\
\delta G &= i\bar{\varepsilon}D_2\gamma_5\chi \,.
\end{aligned}$$

In fact, the variation of S_0 under the this transformation is

$$\begin{aligned}
&\delta S_0 = \\
&= \sum_x \{\bar{\chi}(1 - \frac{a}{2}D_1)^{-1}D_2[-D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon] - \frac{2}{a}\bar{\chi}\varepsilon D_1 A \\
&\qquad + \frac{2i}{a}\bar{\chi}\gamma_5\varepsilon D_1 B + (\bar{\varepsilon}D_2\chi)(1 - \frac{a}{2}D_1)^{-1}F + i(\bar{\varepsilon}D_2\gamma_5\chi)(1 - \frac{a}{2}D_1)^{-1}G\} \,.
\end{aligned}$$

and integrating by part *, this variation becomes

$$\sum_x \{ -\bar{\chi}\varepsilon\big[(1-\frac{a}{2}D_1)^{-1}D_2^2+\frac{2}{a}D_1\big]A+i\bar{\chi}\gamma_5\varepsilon\big[(1-\frac{a}{2}D_1)^{-1}D_2^2+\frac{2}{a}D_1\big]B$$
$$-\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2(F-i\gamma_5 G)\varepsilon+\bar{\chi}D_2\varepsilon(1-\frac{a}{2}D_1)^{-1}F+i\bar{\chi}D_2\gamma_5\varepsilon(1-\frac{a}{2}D_1)^{-1}G\}$$
$$=0\,,$$

where we used the Ginsparg-Wilson relation, which implies

$$(1-\frac{a}{2}D_1)^{-1}D_2^2=-\frac{2}{a}D_1\,.$$

*For instance, for any scalar function $\mathcal{F}$ one has $\mathcal{F}\bar{\varepsilon}D_2\chi=\bar{\chi}D_2\mathcal{F}\varepsilon$.

Failure of the Leibniz rule

The variation of S_{int} under the susy transformation does not vanish because of the failure of the Leibniz rule at finite lattice spacing (Fujikawa 2002 and Dondi and Nicolai 1977)

$$\frac{1}{a}(f(x+a)g(x+a) - f(x)g(x)) =$$
$$= \frac{1}{a}(f(x+a) - f(x))g(x) + \frac{1}{a}f(x)(g(x+a) - g(x))$$
$$+a\frac{1}{a}(f(x+a) - f(x))\frac{1}{a}(g(x+a) - g(x))$$
$$= (\nabla f(x))g(x) + f(x)(\nabla g(x)) + a(\nabla f(x))(\nabla g(x))$$

breaking of supersymmetry is of order $O(a)$.

- In order to discuss the symmetry properties of the lattice Wess-Zumino model one possibility is to modify the action by adding irrelevant terms which make invariant the full action.

- Alternatively, one can modify the supersymmetry transformation in such a way that the action has an exact symmetry for $a \neq 0$.

Since the transformation leaves invariant the free part of the action, this modification must vanish for $g = 0$.

$$\begin{aligned}
&\delta A = \bar{\varepsilon}\chi = \bar{\chi}\varepsilon \\
&\delta B = -i\bar{\varepsilon}\gamma_5\chi = -i\bar{\chi}\gamma_5\varepsilon \\
&\delta\chi = -D_2(A - i\gamma_5 B)\varepsilon - (F - i\gamma_5 G)\varepsilon + gR\varepsilon \\
&\delta F = \bar{\varepsilon}D_2\chi \\
&\delta G = i\bar{\varepsilon}D_2\gamma_5\chi
\end{aligned}$$

- R to be determined by requiring that the variation of the action vanishes.
  - We assume that R depends on the scalar and auxiliary fields and their derivatives and not on χ.

Exact susy transformation for the full action

The variation of the Wess-Zumino action under the transformation is

$$\begin{aligned}\delta S_{WZ} &= \sum_x \{g\bar{\chi}(1-\frac{a}{2}D_1)^{-1}D_2R\varepsilon - m\bar{\chi}[D_2(A-i\gamma_5 B)\varepsilon + (F-i\gamma_5 G)\varepsilon - gR\varepsilon] \\ &\quad + m(A\bar{\varepsilon}D_2\chi + F\bar{\chi}\varepsilon + iB\bar{\varepsilon}D_2\gamma_5\chi - iG\bar{\chi}\gamma_5\varepsilon) + \frac{g}{\sqrt{2}}\bar{\chi}(\bar{\varepsilon}\chi + \gamma_5(\bar{\varepsilon}\gamma_5\chi))\chi \\ &\quad - \sqrt{2}g\bar{\chi}(A+i\gamma_5 B)[D_2(A-i\gamma_5 B)\varepsilon + (F-i\gamma_5 G)\varepsilon - gR\varepsilon] \\ &\quad + \frac{g}{\sqrt{2}}[(A^2-B^2)\bar{\varepsilon}D_2\chi + 2FA\bar{\chi}\varepsilon + 2iFB\bar{\chi}\gamma_5\varepsilon \\ &\quad + 2iAB\bar{\varepsilon}D_2\gamma_5\chi + 2GB\bar{\chi}\varepsilon - 2iGA(\bar{\chi}\gamma_5\varepsilon)]\}\,.\end{aligned}$$

By using the Fierz identity, terms with four fermions cancel as in the continuum.

Moreover, g independent terms cancel out after an integration by part, and one is left with

$$\begin{aligned}\delta S_{WZ} &= \sum_x \{g\bar{\chi}[(1-\frac{a}{2}D_1)^{-1}D_2R + mR]\varepsilon - \frac{g}{\sqrt{2}}[2\bar{\chi}(A+i\gamma_5 B)D_2(A-i\gamma_5 B)\varepsilon \\ &\qquad - \bar{\chi}D_2(A-i\gamma_5 B)^2\varepsilon] + \sqrt{2}g^2\bar{\chi}(A+i\gamma_5 B)R\varepsilon\}\,.\end{aligned}$$

The function R is determined by imposing the vanishing of δS_{WZ}.
By expanding R in powers of g

$$R = R^{(1)} + gR^{(2)} + \cdots$$

and imposing the symmetry condition order by order in perturbation theory, we find

$$R^{(1)} = ((1 - \frac{a}{2}D_1)^{-1}D_2 + m)^{-1}\Delta L$$

with

$$\begin{aligned}\Delta L \equiv& \frac{1}{\sqrt{2}}(2(A + i\gamma_5 B)D_2(A - i\gamma_5 B) - D_2(A - i\gamma_5 B)^2)\\ =& \frac{1}{\sqrt{2}}\{2(AD_2A - BD_2B) - D_2(A^2 - B^2)\\ &+ 2i\gamma_5[(AD_2B + BD_2A) - D_2(AB)]\}\,.\end{aligned}$$

To order g^2 one has

$$R^{(2)} = -\sqrt{2}((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}(A+i\gamma_5 B)((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}\Delta L\,,$$

and for $n \geq 2$

$$R^{(n)} = -\sqrt{2}((1-\frac{a}{2}D_1)^{-1}D_2+m)^{-1}(A+i\gamma_5 B)R^{(n-1)}\,.$$

The formal solution is

$$[(1-\frac{a}{2}D_1)^{-1}D_2+m+\sqrt{2}g(A+i\gamma_5 B)]R = \Delta L\,.$$

- $R \to 0$ for $a \to 0$, since ΔL vanishes in this limit.
- ΔL is different from zero because of the breaking of the Leibniz rule for a finite lattice spacing.

The algebra

By the commutator of two supersymmetries one find a transformation which is still a symmetry of the Wess-Zumino action, i.e. the transformations of the fields form a closed algebra, order by order in g.

Up to order g^1, (the rest can be generalized!)

Two supersymmetry transformations on the scalar field A give

$$\begin{aligned}\delta_1\delta_2 A &= \delta_1(\bar{\varepsilon}_2\chi) \\ &= -\bar{\varepsilon}_2[D_2(A - i\gamma_5 B)\varepsilon_1 + (F - i\gamma_5 G)\varepsilon_1 - gR\varepsilon_1]\end{aligned}$$

and their commutator yields

$$[\delta_2, \delta_1]A = -2\bar{\varepsilon}_1 D_2 \varepsilon_2 A + g(\bar{\varepsilon}_1 R \varepsilon_2 - \bar{\varepsilon}_2 R \varepsilon_1)\,.$$

The order g^1 of the second term on the r.h.s. reads

$$\begin{aligned}&g(\bar{\varepsilon}_1 R^{(1)}\varepsilon_2 - \bar{\varepsilon}_2 R^{(1)}\varepsilon_1) = \\ &\sqrt{2}g\bar{\varepsilon}_2 \frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_2(A^2-B^2) - 2(AD_2A - BD_2B)]\varepsilon_1\end{aligned}$$

Finally, the commutator of two supersymmetries on the scalar field A is

$$[\delta_2, \delta_1]A = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}A$$
$$+ \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(A^2-B^2) - 2(AD_{2\mu}A - BD_{2\mu}B)]\}\,.$$

Similarly, the commutators of two supersymmetries on the other fields, up to terms of order g^1, are

$$[\delta_2, \delta_1]B = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}B$$
$$+ \sqrt{2}g\frac{m(1-\frac{a}{2}D_1)}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(AB) - (AD_{2\mu}B + BD_{2\mu}A)]\}\,,$$
$$[\delta_2, \delta_1]F = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}F$$
$$- \frac{g}{\sqrt{2}}\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(A^2-B^2) - 2(AD_{2\mu}A - BD_{2\mu}B)]\}\,,$$
$$[\delta_2, \delta_1]G = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}G$$
$$- \sqrt{2}g\frac{D_2^2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}[D_{2\mu}(AB) - (AD_{2\mu}B + BD_{2\mu}A)]\}$$

and

$$[\delta_2,\delta_1]\chi = -2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\{D_{2\mu}\chi - \frac{g}{\sqrt{2}}\frac{m(1-\frac{a}{2}D_1)-D_2}{m^2(1-\frac{a}{2}D_1)+\frac{2}{a}D_1}\Big(D_2(A-i\gamma_5B)\gamma_\mu\chi + (A+i\gamma_5B)D_2\gamma_\mu\chi - D_2[(A-i\gamma_5B)\gamma_\mu\chi]\Big)\Big\}\,.$$

Therefore, the general expression of these commutators is

$$[\delta_1,\delta_2]\Phi = \alpha^\mu P_\mu^\Phi(\Phi)\,, \qquad \Phi = (A,B,F,G,\chi)\,,$$

where $\alpha^\mu = -2\bar{\varepsilon}_2\gamma^\mu\varepsilon_2$ and $P_\mu^\Phi(\Phi)$ are polynomials in Φ defined as

$$P_\mu^\Phi(\Phi) = D_{2\mu}\Phi + O(g)$$

We have verified that the closure works, i.e. the action is invariant under the transformation (up to terms of order g^1).

$$\Phi \to \Phi + \alpha^\mu P_\mu^\Phi(\Phi)$$

Notice that, in the continuum limit $D_{2\mu} \to \partial_\mu$ and the transformation reduces to

$$\Phi \to \Phi + \alpha^\mu \partial_\mu \Phi$$

Conclusions and Outlook

- The Wess-Zumino model is an interesting model to understand how to put GW fermions with exact lattice supersymmetry.
- Study of the Ward-Takahashi identities.
- numerical simulations of this model (at leat in two dimensions)
- The forward step would be to apply to $N = 1$ SYM (more tricky!)
- $\cdots$