Formulation of chiral gauge theories

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- Starting from basic structure of
  Narayanan, Neuberger 1993, 1994, 1995
  and Lüscher 1999, 2000
  recently generalization W.K. 2003

- In view of Ginsparg-Wilson (GW) Dirac operator $D = \rho(\mathbf{1} - V)$ with $V^{-1} = V^{\dagger}$
  in W.K. 2003 more generally $D = F(V)$
  and chiral projections $P_{-}(V)$ and $\bar{P}_{+}(V)$

- Here formulation without reference to V
  to reveal truly relevant features,
  restrictions on spectra of D removed,
  much more general structure of P_{-} and $\bar{P}_{+}$

- Also condition on basis transformations
  refined and related properties of
  equivalence classes of bases worked out

- Topics:
  Dirac operators, spectral representations
  structure of chiral projections
  forms of correlation functions
  equivalence classes of bases
  transformation properties
  variations, perturbation theory

- For $[D^\dagger, D] = 0$ and $D^\dagger = \gamma_5 D \gamma_5$ we have the spectral representation

$$D = \sum_j \hat{\lambda}_j (P_j^+ + P_j^-) + \sum_k (\lambda_k P_k^{\mathbf{I}} + \lambda_k^* P_k^{\mathbf{II}})$$

  with Im $\hat{\lambda}_j = 0$ and Im $\lambda_k > 0$ and where $\gamma_5 P_j^\pm = P_j^\pm \gamma_5 = \pm P_j^\pm$ and $\gamma_5 P_k^{\mathbf{I}} = P_k^{\mathbf{II}} \gamma_5$

- Since Tr$(\gamma_5 \mathbf{1})$ = Tr$(\gamma_5 P_k^{\mathbf{I}})$ = Tr$(\gamma_5 P_k^{\mathbf{II}}) = 0$ we get for $N_j^\pm = \text{Tr}\, P_j^\pm$

$$\sum_j (N_j^+ - N_j^-) = 0$$

  With $\hat{\lambda}_0 = 0$ the index is $I = N_0^+ - N_0^-$

- In contrast to the formulations with V
  - not restricted to one real eigenvalue in addition to zero
  - for complex ones now different moduli for the same phase possible
  - on the other hand, realizations relying on V no longer applicable, other ones to be used

- To get particular realizations we require

$$\tfrac{1}{2}(D + D^\dagger) = DD^\dagger\, F\big(DD^\dagger, \tfrac{1}{2}(D + D^\dagger)\big)$$

  where F is a nonsingular function
  which is local for local D

- In W.K. 2002 and Fujikawa et. al 2002 special cases depending on $DD^\dagger$ only and with monotony, thus only one real eigenvalue in addition to zero

- Example $F = \sum_{\nu=0}^{M} \mathcal{C}_\nu (DD^\dagger)^\nu$ of W.K. 2002 can readily be extended to case here with up to $2M+1$ further real eigenvalues

- Evaluation possible by extension of method of chirally improved fermions by Gattringer et. al 2001 (with systematic expansion of Dirac operator)

  Mapping of GW equation to system of coupled equations there can as well be done in case of more general relation here

- For chiral projections P_- and $\bar{P}_+$ required

$$\bar{P}_+ D = D P_-$$

- Implies $[P_-, DD^\dagger] = [\bar{P}_+, DD^\dagger] = 0$ for

$$DD^\dagger = \sum_j \hat{\lambda}_j^2 (P_j^+ + P_j^-) + \sum_k |\lambda_k|^2 (P_k^{\mathrm{I}} + P_k^{\mathrm{II}})$$

so that P_- and $\bar{P}_+$ decompose as

$$P_- = \sum_j P_j^{\mathsf{X}} + \sum_k P_k^{\mathsf{R}}, \quad \bar{P}_+ = \sum_j \bar{P}_j^{\mathsf{X}} + \sum_k \bar{P}_k^{\mathsf{R}}$$

P_j^{X} and $\bar{P}_j^{\mathsf{X}}$ in subspace of $P_j^+ + P_j^-$
P_k^{R} and $\bar{P}_k^{\mathsf{R}}$ in subspace of $P_k^{\mathrm{I}} + P_k^{\mathrm{II}}$

- Condition $\bar{P}_+ D = D P_-$ and spectral representation of D used to determine these projections

- Expressing P_k^{R} and $\bar{P}_k^{\mathsf{R}}$ by P_k^{I} and P_k^{II} we get

$$\begin{aligned} P_k^{\mathsf{R}} &= c_k P_k^{\mathrm{I}} + (1-c_k) P_k^{\mathrm{II}} \\ &\qquad - \sqrt{c_k(1-c_k)}\,\gamma_5 (\mathrm{e}^{i\varphi_k} P_k^{\mathrm{I}} + \mathrm{e}^{-i\varphi_k} P_k^{\mathrm{II}}) \\ \bar{P}_k^{\mathsf{R}} &= c_k P_k^{\mathrm{I}} + (1-c_k) P_k^{\mathrm{II}} \\ &\qquad + \sqrt{c_k(1-c_k)}\,\gamma_5 \big(\mathrm{e}^{-i\bar{\varphi}_k} P_k^{\mathrm{I}} + \mathrm{e}^{i\bar{\varphi}_k} P_k^{\mathrm{II}}\big) \end{aligned}$$

$0 \le c_k \le 1, \quad \mathrm{e}^{i(\varphi_k + \bar{\varphi}_k - 2\alpha_k)} = -1, \quad \mathrm{e}^{i\alpha_k} = \lambda_k / |\lambda_k|$
with the relations

$$\mathrm{Tr}\, P_k^{\mathsf{R}} = \mathrm{Tr}\, \bar{P}_k^{\mathsf{R}} = \mathrm{Tr}\, P_k^{\mathrm{I}} = \mathrm{Tr}\, P_k^{\mathrm{II}} =: \tilde{N}_k$$

- For $j \neq 0$ we obtain $P_j^\times = \bar{P}_j^\times$ so that for
$$\bar{N} = \mathrm{Tr}\,\bar{P}_+, \qquad N = \mathrm{Tr}\,P_-$$
we have $\bar{N} - N = \mathrm{Tr}\,\bar{P}_0^\times - \mathrm{Tr}\,P_0^\times$. Thus choice
$$\bar{P}_0^\times = P_0^+, \qquad P_0^\times = P_0^-$$
which leads to $\bar{N} - N = I$

- For $I = 0$ it follows that $\sum_{j\neq 0} N_j^+ = \sum_{j \neq 0} N_j^-$ so that to get $\mathrm{Tr}\,\mathbf{1} =: 2d$ for
$$\bar{N} + N = N_0^+ + N_0^- + 2\sum_{j\neq 0} \mathrm{Tr}\,P_j^\times + 2\sum_k \tilde{N}_k$$
we must put
$$P_j^\times = P_j^+ \qquad \textbf{or} \qquad P_j^\times = P_j^-$$

- For general I we then have
$$\bar{N} = d,\;\; N = d - I \qquad \textbf{or} \qquad \bar{N} = d + I,\;\; N = d$$

- For the dimensions in the decompositions of the chiral projections P_- and $\bar{P}_+$ we thus obtain
$$N = N_0^- + L, \qquad \bar{N} = N_0^+ + L$$
$$L = \sum_{j \neq 0} N_j^\pm + \sum_k \tilde{N}_k$$

- Chiral projections may also be expressed by

$$P_- = \frac{1}{2}(\mathbf{1} - \gamma_5 G), \qquad \bar{P}_+ = \frac{1}{2}(\mathbf{1} + \bar{G}\gamma_5)$$

- G and $\bar{G}$ are unitary and γ_5-Hermitian with

$$G = P_0^+ + P_0^- \mp \sum_{j\neq 0} \left(P_j^+ + P_j^-\right) + \sum_k \left(\mathrm{e}^{i\phi_k} P_k^{\mathsf{A}} + \mathrm{e}^{-i\phi_k} P_k^{\mathsf{B}}\right)$$
$$\bar{G} = P_0^+ + P_0^- \pm \sum_{j\neq 0} \left(P_j^+ + P_j^-\right) + \sum_k \left(\mathrm{e}^{i\bar{\phi}_k} \bar{P}_k^{\mathsf{A}} + \mathrm{e}^{-i\bar{\phi}_k} \bar{P}_k^{\mathsf{B}}\right)$$

related to the quantities introduced before by

$$P_k^{\mathsf{A}} = \left(h_k^2 P_k^{\mathrm{I}} + b_k^2 P_k^{\mathrm{II}} - i b_k h_k \gamma_5 (P_k^{\mathrm{I}} - P_k^{\mathrm{II}})\right)/(h_k^2 + b_k^2)$$
$$P_k^{\mathsf{B}} = \left(b_k^2 P_k^{\mathrm{I}} + h_k^2 P_k^{\mathrm{II}} + i b_k h_k \gamma_5 (P_k^{\mathrm{I}} - P_k^{\mathrm{II}})\right)/(h_k^2 + b_k^2)$$

$$h_k = a_k \sin\varphi_k + \sin\phi_k, \quad b_k = 1 - 2c_k, \quad a_k = 2\sqrt{c_k(1-c_k)}$$

$$\cos\phi_k = a_k \cos\varphi_k, \qquad \sin\phi_k = \sqrt{1 - a_k^2 \cos^2\varphi_k}$$

for G and by analogous relations for $\bar{G}$

- Beause of the opposite signs of the j-sums (which to allow for a non-vanishing index must not vanish) obvious that generally $\bar{G} \neq G$

- Special cases $G = \mathbf{1}$, $\bar{G} \neq \mathbf{1}$ and $\bar{G} = \mathbf{1}$, $G \neq \mathbf{1}$

- General relation

$$D + \bar{G} D^\dagger G = 0$$

- Choosing $c_k = \frac{1}{2}$ in chiral projections
  G and $\bar{G}$ commute with D

- In case $c_k = \frac{1}{2}$ for the mentioned realization of D

  $$V = \mathbf{1} - 2D\, F\big(DD^\dagger, \tfrac{1}{2}(D + D^\dagger)\big)$$

  and $\bar{G}G = V$

- In GW case introduced by P. Hasenfratz 2002

  $$G = \big((1-s)\mathbf{1} + sV\big)/\mathcal{N}, \quad \bar{G} = \big(s\mathbf{1} + (1-s)V\big)/\mathcal{N}$$

  with $\mathcal{N} = \sqrt{\mathbf{1} - 2s(1-s)\big(\mathbf{1} - \frac{1}{2}(V + V^\dagger)\big)}$ and real parameter $s \neq \frac{1}{2}$, for which $\bar{G}G = V$,
  also realization for more general operators here

- Special choice $G = V$, $\bar{G} = \mathbf{1}$
  in Narayanan, Neuberger 1993, 1994, 1995
  and Lüscher 1999, 2000 with V of GW case,
  in particular V of Neuberger 1998

- Non-vanishing fermionic correlation functions are

$$\langle \psi_{\sigma_{r+1}} \dots \psi_{\sigma_N} \bar{\psi}_{\bar{\sigma}_{r+1}} \dots \bar{\psi}_{\bar{\sigma}_{\bar{N}}} \rangle_{\mathsf{f}} = \frac{1}{r!} \sum_{\bar{\sigma}_1 \dots \bar{\sigma}_r} \sum_{\sigma_1, \dots, \sigma_r} \bar{\Upsilon}^*_{\bar{\sigma}_1 \dots \bar{\sigma}_{\bar{N}}} \Upsilon_{\sigma_1 \dots \sigma_N} D_{\bar{\sigma}_1 \sigma_1} \dots D_{\bar{\sigma}_r \sigma_r}$$

with the alternating multilinear forms

$$\Upsilon_{\sigma_1 \dots \sigma_N} = \sum_{i_1, \dots, i_N = 1}^{N} \epsilon_{i_1, \dots, i_N} u_{\sigma_1 i_1} \dots u_{\sigma_N i_N}$$

$$\bar{\Upsilon}_{\bar{\sigma}_1 \dots \bar{\sigma}_{\bar{N}}} = \sum_{j_1, \dots j_{\bar{N}} = 1}^{\bar{N}} \epsilon_{j_1, \dots, j_{\bar{N}}} \bar{u}_{\bar{\sigma}_1 j_1} \dots \bar{u}_{\bar{\sigma}_{\bar{N}} j_{\bar{N}}}$$

where the bases $\bar{u}_{\bar{\sigma} j}$ and $u_{\sigma i}$ satisfy

$$P_- = u u^\dagger, \quad u^\dagger u = \mathbf{1}_{\mathsf{w}}, \qquad \bar{P}_+ = \bar{u} \bar{u}^\dagger, \quad \bar{u}^\dagger \bar{u} = \mathbf{1}_{\bar{\mathsf{w}}}$$

- Note that averages of $|\Upsilon_{\sigma_1 \dots \sigma_N}|^2$ and $|\bar{\Upsilon}_{\bar{\sigma}_1 \dots \bar{\sigma}_{\bar{N}}}|^2$ equal to one for any N and $\bar{N}$ since

$$\frac{1}{N!} \sum_{\sigma_1, \dots, \sigma_N} |\Upsilon_{\sigma_1 \dots \sigma_N}|^2 = \frac{1}{\bar{N}!} \sum_{\bar{\sigma}_1, \dots, \bar{\sigma}_{\bar{N}}} |\bar{\Upsilon}_{\bar{\sigma}_1 \dots \bar{\sigma}_{\bar{N}}}|^2 = 1$$

- While P_- and $\bar{P}_+$ invariant under unitary basis transformations $u^{(S)} = uS$, $\bar{u}^{(\bar{S})} = \bar{u}\bar{S}$, $\Upsilon_{\sigma_1 \dots \sigma_N}$ and $\bar{\Upsilon}_{\bar{\sigma}_1 \dots \bar{\sigma}_{\bar{N}}}$ multiplied by $\det_{\mathsf{w}} S$ and $\det_{\bar{\mathsf{w}}} \bar{S}$, therefore condition

$$\det{}_{\mathsf{w}} S \cdot \det{}_{\bar{\mathsf{w}}} \bar{S}^\dagger = 1$$

(constant equal to one in order that invariance also for general linear combinations of functions)

- Total set of pairs of bases because of condition

$$\mathrm{det}_{\mathrm{w}} S \cdot \mathrm{det}_{\bar{\mathrm{w}}} \bar{S}^{\dagger} = 1$$

  decomposes into equivalence classes
  of which **one** is to be chosen

  - Without condition all bases
    related to one of the chiral projections
    connected by unitary transformations
  - With condition total set of pairs of bases
    u and $\bar{u}$ decomposes into inequivalent subsets
    which constitute equivalence classes
  - Transformations which respect condition
    do not connect beyond equivalence class

- Different equivalence classes are related by
  pairs of basis transformations for which

$$\mathrm{det}_{\mathrm{w}} S \cdot \mathrm{det}_{\bar{\mathrm{w}}} \bar{S}^{\dagger} = \mathrm{e}^{i\Theta}, \quad \Theta \neq 0$$

  The phase factor $\mathrm{e}^{i\Theta}$ describes how the results
  of the respective formulations of the theory differ

- Considerations simplify by noting that S
  with $\det S = \mathrm{e}^{i\theta}$ can be expressed by product
  of irrelevant unimodular matrix $S\mathrm{e}^{-i\theta/N}$
  and matrix $\mathrm{e}^{i\theta/N}\mathbf{1}$

- Properties of chiral projections imply
  (1) corresponding decomposition of bases
  (2) relations between u and $\bar{u}$
  - From the general relation

    $$\bar{P}_k^{\mathsf{R}} = |\lambda_k|^{-2} D P_k^{\mathsf{R}} D^\dagger$$

    putting $P_k^{\mathsf{R}} = \sum_{l=1}^{N_k} u_l^{[k]} u_l^{[k]\dagger}$ we get

    $$\bar{u}_l^{[k]} = \mathrm{e}^{-i\Theta_k} |\lambda_k|^{-1} D u_l^{[k]}$$

    with phases Θ_k so that $\bar{P}_k^{\mathsf{R}} = \sum_{l=1}^{N_k} \bar{u}_l^{[k]} \bar{u}_l^{[k]\dagger}$
  - For $P_j^{\pm} = \sum_{l=1}^{N_j^{\pm}} u_l^{\pm[j]} u_l^{\pm[j]\dagger} = \sum_{l=1}^{N_j^{\pm}} \bar{u}_l^{\pm[j]} \bar{u}_l^{\pm[j]\dagger}$
    where $j \neq 0$ we have with phases $\Theta_j^{\pm}$

    $$\bar{u}_l^{\pm[j]} = \mathrm{e}^{-i\Theta_j^{\pm}} |\hat{\lambda}_j|^{-1} D u_l^{\pm[j]}$$

- Because of $N - N_0^- = \bar{N} - N_0^+ = L$
  so far diagonal $L \times L$ submatrix M_L of $\bar{u}^\dagger D u$
  in which eigenvalues

  $$\mathrm{e}^{i\Theta_k} |\lambda_k|, \qquad \mathrm{e}^{i\Theta_j^{\pm}} |\hat{\lambda}_j|$$

  have multiplicities N_k and $N_j^{\pm}$ respectively

- Zero-mode part described by
  $P_0^- = \sum_{l=1}^{N_0^-} u_l u_l^\dagger$ and $P_0^+ = \sum_{l=1}^{N_0^+} \bar{u}_l \bar{u}_l^\dagger$
  with bases satisfying $D u_l = 0$ and $D \bar{u}_l = 0$

- With this we find for the correlation functions

$$\langle \psi_{\sigma_{r+1}} \dots \psi_{\sigma_N} \bar{\psi}_{\bar{\sigma}_{r+1}} \dots \bar{\psi}_{\bar{\sigma}_{\bar{N}}} \rangle_{\mathsf{f}} = \sum_{\sigma'_{r+1},\dots,\sigma'_N} \epsilon^{\sigma'_{r+1}\dots\sigma'_N}_{\sigma_{r+1}\dots\sigma_N} \sum_{\bar{\sigma}'_{r+1},\dots,\bar{\sigma}'_{\bar{N}}} \epsilon^{\bar{\sigma}'_{r+1}\dots\bar{\sigma}'_{\bar{N}}}_{\bar{\sigma}_{r+1}\dots\bar{\sigma}_{\bar{N}}} \frac{1}{(L-r)!} \mathcal{G}_{\sigma'_{r+1}\bar{\sigma}'_{r+1}} \dots \mathcal{G}_{\sigma'_L\bar{\sigma}'_L}$$

$$\mathrm{e}^{-i\Theta_{\mathrm{z}}^-} u_{\sigma_{L+1},L+1} \dots u_{\sigma_N N} \quad \mathrm{e}^{i\Theta_{\mathrm{z}}^+} \bar{u}^\dagger_{L+1,\bar{\sigma}_{L+1}} \dots \bar{u}^\dagger_{\bar{N}\bar{\sigma}_{\bar{N}}} \ \det M_L$$

where $\mathcal{G} = \bar{P}_- \bar{D}^{-1} \bar{\bar{P}}_+$, with $\bar{D}$, $\bar{P}_-$, $\bar{\bar{P}}_+$ operators D, P_-, $\bar{P}_+$ restricted to the subspace on which $\mathbf{1} - P_0^+ - P_0^-$ projects,

$$\det M_L = \prod_{j\neq 0} (\mathrm{e}^{i\Theta_j^\pm} |\hat{\lambda}_j|)^{N_j^\pm} \prod_k (\mathrm{e}^{i\Theta_k} |\lambda_k|)^{\tilde{N}_k}$$

and phases Θ_{z}^+ and Θ_{z}^- related to zero modes

- Note that basis dependence of $\det M_L$ in subspace and that additional one by u_l and $\bar{u}_l^\dagger$ of zero modes

- Equivalence class of bases characterized by value

$$\sum_k N_k \Theta_k + \sum_{j\neq 0} N_j^\mp \Theta_j^\mp + \Theta_{\mathrm{z}}^+ - \Theta_{\mathrm{z}}^-$$

- Conditions for equivalence class uS, $\bar{u}S$

$$P_- = uu^\dagger, \quad u^\dagger u = \mathbf{1}_{\mathsf{w}}, \qquad \bar{P}_+ = \bar{u}\bar{u}^\dagger, \quad \bar{u}^\dagger\bar{u} = \mathbf{1}_{\bar{\mathsf{w}}}$$

$$\mathrm{det}_{\mathsf{w}} S \cdot \mathrm{det}_{\bar{\mathsf{w}}} \bar{S}^\dagger = 1$$

- Gauge transformations $P'_- = \mathcal{T}P_-\mathcal{T}^\dagger$, $\bar{P}'_+ = \mathcal{T}\bar{P}_+\mathcal{T}^\dagger$ for $G \neq \mathbf{1}$, $\bar{G} \neq \mathbf{1}$ with $[\mathcal{T},P_-] \neq 0$, $[\mathcal{T},\bar{P}_+] \neq 0$ generally imply for transformed equivalence class

$$u'S' = \mathcal{T}uSS_{\mathcal{T}}, \qquad \bar{u}'\bar{S}' = \mathcal{T}\bar{u}\bar{S}\bar{S}_{\mathcal{T}}$$

with u', $\bar{u}'$, S', $\bar{S}'$ satisfying transformed conditions and unitary $S_{\mathcal{T}} = \mathcal{S}(\mathcal{T},\mathcal{U})$ and $\bar{S}_{\mathcal{T}} = \bar{\mathcal{S}}(\mathcal{T},\mathcal{U})$ for which $\mathrm{det}_{\mathsf{w}}\mathcal{S}(\mathbf{1},\mathcal{U})\big(\mathrm{det}_{\bar{\mathsf{w}}}\bar{\mathcal{S}}(\mathbf{1},\mathcal{U})\big)^* = 1$ and
$\mathrm{det}_{\mathsf{w}}(\mathcal{S}(\mathcal{T}_{\mathsf{a}},\mathcal{U})\mathcal{S}(\mathcal{T}_{\mathsf{b}},\mathcal{T}_{\mathsf{a}}\mathcal{U}\mathcal{T}_{\mathsf{a}}^\dagger))\big(\mathrm{det}_{\bar{\mathsf{w}}}(\bar{\mathcal{S}}(\mathcal{T}_{\mathsf{a}},\mathcal{U})\bar{\mathcal{S}}(\mathcal{T}_{\mathsf{b}},\mathcal{T}_{\mathsf{a}}\mathcal{U}\mathcal{T}_{\mathsf{a}}^\dagger))\big)^*$
$= \mathrm{det}_{\mathsf{w}}\mathcal{S}(\mathcal{T}_{\mathsf{b}}\mathcal{T}_{\mathsf{a}},\mathcal{T}_{\mathsf{b}}\mathcal{T}_{\mathsf{a}}\mathcal{U}\mathcal{T}_{\mathsf{a}}^\dagger\mathcal{T}_{\mathsf{b}}^\dagger)\big(\mathrm{det}_{\bar{\mathsf{w}}}\bar{\mathcal{S}}(\mathcal{T}_{\mathsf{b}}\mathcal{T}_{\mathsf{a}},\mathcal{T}_{\mathsf{b}}\mathcal{T}_{\mathsf{a}}\mathcal{U}\mathcal{T}_{\mathsf{a}}^\dagger\mathcal{T}_{\mathsf{b}}^\dagger)\big)^*$

- Insertion gives for correlation functions

$$\langle\psi'_{\sigma'_1}\ldots\psi'_{\sigma'_R}\bar{\psi}'_{\bar{\sigma}'_1}\ldots\bar{\psi}'_{\bar{\sigma}'_{\bar{R}}}\rangle'_{\mathsf{f}} = \mathrm{e}^{i\vartheta_{\mathcal{T}}}\sum_{\sigma_1,\ldots,\sigma_R}\sum_{\bar{\sigma}_1,\ldots,\bar{\sigma}_{\bar{R}}}\mathcal{T}_{\sigma'_1\sigma_1}\ldots$$
$$\ldots\mathcal{T}_{\sigma'_R\sigma_R}\langle\psi_{\sigma_1}\ldots\psi_{\sigma_R}\bar{\psi}_{\bar{\sigma}_1}\ldots\bar{\psi}_{\bar{\sigma}_{\bar{R}}}\rangle_{\mathsf{f}}\,\mathcal{T}^\dagger_{\bar{\sigma}_1\bar{\sigma}'_1}\ldots\mathcal{T}^\dagger_{\bar{\sigma}_{\bar{R}}\bar{\sigma}'_{\bar{R}}}$$

where $\mathrm{e}^{i\vartheta_{\mathcal{T}}} = \mathrm{det}_{\mathsf{w}}S_{\mathcal{T}} \cdot \mathrm{det}_{\bar{\mathsf{w}}}\bar{S}^\dagger_{\mathcal{T}}$ with $\vartheta_{\mathbf{1}} = 0$

- Factor $\mathrm{det}_{\mathsf{w}}S_{\mathcal{T}} \cdot \mathrm{det}_{\bar{\mathsf{w}}}\bar{S}^\dagger_{\mathcal{T}} = \mathrm{e}^{i\vartheta_{\mathcal{T}}}$ for $\vartheta_{\mathcal{T}} \neq 0$ just form met for transformations to inequivalent bases:

  This part of transformation separated off by requiring $\vartheta_{\mathcal{T}} = 0$

- In special case $G \neq \mathbf{1}$, $\bar{G} = \mathbf{1}$ because of $\bar{P}'_+ = \bar{P}_+$ no change of equivalence class part $\bar{u}\bar{S}$

- Then, however, because of $[\mathcal{T}, \bar{P}_+] = 0$ possible to rewrite

$$\bar{u}\bar{S} = \mathcal{T}\bar{u}\bar{S}\widehat{S}_{\mathcal{T}}$$

and to calculate

$$\mathrm{det}_{\bar{\mathrm{w}}}\widehat{S}^{\dagger}_{\mathcal{T}} = \exp(\tfrac{1}{2}\mathrm{Tr}\,\mathcal{B})$$

where $\mathcal{T} = \exp(\mathcal{B})$ (and $\mathrm{Tr}\,\mathcal{B} = 4i \sum_{n,\ell} b_n^{\ell}\, \mathrm{tr}_{\mathrm{g}} T^{\ell}$)

- Then factor in correlation functions

$$\mathrm{e}^{i\vartheta_{\mathcal{T}}} = \mathrm{det}_{\mathrm{w}} S_{\mathcal{T}} \cdot \exp(\tfrac{1}{2}\mathrm{Tr}\,\mathcal{B})$$

where now $\mathrm{det}_{\mathrm{w}} S_{\mathcal{T}}$ form of transformation to inequivalent bases:

Separating off this part one here remains with $\vartheta_{\mathcal{T}} = -i \exp(\mathrm{Tr}\,\mathcal{B})$

- CP transformations of Dirac operator

$$D(\mathcal{U}^{\mathsf{CP}}) = \mathcal{W} D^{\mathsf{T}}(\mathcal{U}) \mathcal{W}^{\dagger}, \quad \mathcal{W} = \mathcal{P}\gamma_4 C^{\dagger}$$

  with $\mathcal{P}_{n'n} = \delta^4_{n'\tilde{n}}$, $U^{\mathsf{CP}}_{4n} = U^*_{4\tilde{n}}$ and $U^{\mathsf{CP}}_{kn} = U^*_{k,\tilde{n}-\hat{k}}$ for $k = 1, 2, 3$, where $\tilde{n} = (-\vec{n}, n_4)$, and C charge conjugation matrix

- Implies for P_- and $\bar{P}_+$ the relations

$$P_-^{\mathsf{CP}}(\mathcal{U}^{\mathsf{CP}}) = \mathcal{W} \bar{P}_+^{\mathsf{T}}(\mathcal{U}) \mathcal{W}^{\dagger}, \ \ \bar{P}_+^{\mathsf{CP}}(\mathcal{U}^{\mathsf{CP}}) = \mathcal{W} P_-^{\mathsf{T}}(\mathcal{U}) \mathcal{W}^{\dagger}$$

  which give for the index $I^{\mathsf{CP}} = -I$

- Untransformed and transformed forms

$$P_-(\mathcal{U}) = \frac{1}{2}(\mathbf{1} - \gamma_5 G(\mathcal{U})), \qquad \bar{P}_+(\mathcal{U}) = \frac{1}{2}(\mathbf{1} + \bar{G}(\mathcal{U})\gamma_5)$$

$$P_-^{\mathsf{CP}}(\mathcal{U}^{\mathsf{CP}}) = \frac{1}{2}\big(\mathbf{1} - \gamma_5 \bar{G}(\mathcal{U}^{\mathsf{CP}})\big), \quad \bar{P}_+^{\mathsf{CP}}(\mathcal{U}^{\mathsf{CP}}) = \frac{1}{2}\big(\mathbf{1} + G(\mathcal{U}^{\mathsf{CP}})\gamma_5\big)$$

  obviously differ by interchange of G and $\bar{G}$,
  thus because generally $\bar{G} \neq G$
  not symmetric situation of continuum

- With basic conditions satisfied by u, $\bar{u}$, S, $\bar{S}$ as well as by u^{CP}, $\bar{u}^{\mathsf{CP}}$, S^{CP}, $\bar{S}^{\mathsf{CP}}$

$$u^{\mathsf{CP}} S^{\mathsf{CP}} = \mathcal{W} \bar{u}^* \bar{S}^* S_\zeta, \quad \bar{u}^{\mathsf{CP}} \bar{S}^{\mathsf{CP}} = \mathcal{W} u^* S^* \bar{S}_\zeta$$

  where S_ζ and $\bar{S}_\zeta$ unitary operators

- Then for correlation functions

$$\langle \psi^{\mathsf{CP}}_{\sigma'_1} \dots \psi^{\mathsf{CP}}_{\sigma'_R} \bar{\psi}^{\mathsf{CP}}_{\bar{\sigma}'_1} \dots \bar{\psi}^{\mathsf{CP}}_{\bar{\sigma}'_{\bar{R}}} \rangle^{\mathsf{CP}}_{\mathsf{f}} = \mathrm{e}^{i\vartheta_{\mathsf{CP}}} \sum_{\sigma_1,\dots,\sigma_R} \sum_{\bar{\sigma}_1,\dots,\bar{\sigma}_{\bar{R}}} \mathcal{W}^\dagger_{\bar{\sigma}_1 \bar{\sigma}'_1} \dots$$
$$\dots \mathcal{W}^\dagger_{\bar{\sigma}_{\bar{R}} \bar{\sigma}'_{\bar{R}}} \langle \psi_{\bar{\sigma}_1} \dots \psi_{\bar{\sigma}_{\bar{R}}} \bar{\psi}_{\sigma_1} \dots \bar{\psi}_{\sigma_R} \rangle_{\mathsf{f}} \mathcal{W}_{\sigma'_1 \sigma_1} \dots \mathcal{W}_{\sigma'_R \sigma_R}$$

  where $\mathrm{e}^{i\vartheta_{\mathsf{CP}}} = \det_{\bar{\mathsf{w}}} S_\zeta \cdot \det_{\mathsf{w}} \bar{S}^\dagger_\zeta$

- Repetition of transformation must lead back, therefore choice of S_ζ and $\bar{S}_\zeta$ restricted such that $\vartheta_{\mathsf{CP}} = -i \ln(\det_{\bar{\mathsf{w}}} S_\zeta \cdot \det_{\mathsf{w}} \bar{S}^\dagger_\zeta\ )$ same value

- Factor $\det_{\bar{\mathsf{w}}} S_\zeta \cdot \det_{\mathsf{w}} \bar{S}^\dagger_\zeta = \mathrm{e}^{i\vartheta_{\mathsf{CP}}}$ again form met for transformations to inequivalent bases and this part separated off for $\vartheta_{\mathsf{CP}} = 0$

- We define gauge-field variations by

$$\delta\phi(\mathcal{U}) = \left.\frac{\mathrm{d}\,\phi\big(\mathcal{U}(t)\big)}{\mathrm{d}\,t}\right|_{t=0}, \qquad \mathcal{U}_\mu(t) = \mathrm{e}^{t\mathcal{B}_\mu^{\text{left}}}\mathcal{U}_\mu \mathrm{e}^{-t\mathcal{B}_\mu^{\text{right}}}$$

  where $(\mathcal{U}_\mu)_{n'n} = U_{\mu n}\delta^4_{n',n+\hat{\mu}}$ and $(\mathcal{B}_\mu^{\text{l/r}})_{n'n} = B_{\mu n}^{\text{l/r}}\delta^4_{n',n}$

- In special case of gauge transformations then

$$\mathcal{B}_\mu^{\text{left}} = \mathcal{B}_\mu^{\text{right}} = \mathcal{B}$$

- Varying logarithm of condition $\det_{\mathrm{w}} S \cdot \det_{\bar{\mathrm{w}}} \bar{S}^\dagger = 1$

$$\mathrm{Tr}_{\bar{\mathrm{w}}}(\bar{S}^\dagger \delta \bar{S}) - \mathrm{Tr}_{\mathrm{w}}(S^\dagger \delta S) = 0$$

  which with $u^{(S)} = Su$ and $\bar{u}^{(\bar{S})} = \bar{S}\bar{u}$ becomes

$$\mathrm{Tr}(\delta\bar{u}^{(\bar{S})}u^{(\bar{S})\dagger}) - \mathrm{Tr}(\delta u^{(S)}u^{(S)\dagger}) = \mathrm{Tr}(\delta\bar{u}\,\bar{u}^\dagger) - \mathrm{Tr}(\delta u\, u^\dagger)$$

- Solely using the basic conditions

$$P_- = uu^\dagger, \quad u^\dagger u = \mathbf{1}_{\mathrm{w}}, \qquad \bar{P}_+ = \bar{u}\bar{u}^\dagger, \quad \bar{u}^\dagger\bar{u} = \mathbf{1}_{\bar{\mathrm{w}}}$$

  many variational relations (weaker conditions), for example

$$\mathrm{Tr}\big(P_-[\delta_1 P_-, \delta_2 P_-]\big) = \\ \delta_1 \mathrm{Tr}(\delta_2 u\, u^\dagger) - \delta_2 \mathrm{Tr}(\delta_1 u\, u^\dagger) + \mathrm{Tr}(\delta_{[2,1]} u\, u^\dagger)$$

  holding for general generators $\mathcal{B}_{\mu(1)}^{\text{left}}$, $\mathcal{B}_{\mu(1)}^{\text{right}}$ and $\mathcal{B}_{\mu(2)}^{\text{left}}$, $\mathcal{B}_{\mu(2)}^{\text{right}}$ and $[\mathcal{B}_{\mu(2)}^{\text{left}}, \mathcal{B}_{\mu(1)}^{\text{left}}]$, $[\mathcal{B}_{\mu(2)}^{\text{right}}, \mathcal{B}_{\mu(1)}^{\text{right}}]$

- Effective action **only** in absence of zero modes of D and thus also only for $I = 0$, then

$$-\delta\,\mathcal{S}_{\text{eff}} = \text{Tr}(P_- D^{-1}\delta D) + \text{Tr}(\delta u\, u^\dagger) - \text{Tr}(\delta \bar{u}\, \bar{u}^\dagger)$$

- For gauge transformations with $\mathcal{T}(t) = \text{e}^{t\mathcal{B}}$

  - $\delta^{\text{G}}\mathcal{O} = [\mathcal{B}, \mathcal{O}]$
    from $\mathcal{O}(\mathcal{U}(t)) = \mathcal{T}(t)\,\mathcal{O}(\mathcal{U}(0))\,\mathcal{T}^\dagger(t)$

  - $\delta^{\text{G}} u = \mathcal{B}\, u + u\, S_{\text{x}}^\dagger\, \delta^{\text{G}} S_{\text{x}}$
    from $u(t) = \mathcal{T}(t) u(0) S_{\text{x}}(t)$ for $[\mathcal{T}, P_-] \neq 0$

  so that

$$\text{Tr}(P_- D^{-1}\delta^{\text{G}} D) = \text{Tr}(\mathcal{B}\bar{P}_+) - \text{Tr}(\mathcal{B}P_-)$$

$$\text{Tr}(\delta^{\text{G}} u\, u^\dagger) = \text{Tr}(\mathcal{B}P_-) + \text{Tr}_{\text{w}}(S_{\text{x}}^\dagger\, \delta^{\text{G}} S_{\text{x}})$$

- For $[\mathcal{T}, P_-] \neq 0$, $[\mathcal{T}, \bar{P}_+] \neq 0$ then

$$\delta^{\text{G}}\,\mathcal{S}_{\text{eff}} = \text{Tr}_{\text{w}}(S_{\mathcal{T}}^\dagger\, \delta^{\text{G}} S_{\mathcal{T}}) - \text{Tr}_{\bar{\text{w}}}(\bar{S}_{\mathcal{T}}^\dagger\, \delta^{\text{G}} \bar{S}_{\mathcal{T}}) = i\,\delta^{\text{G}}\vartheta_{\mathcal{T}}$$

  since $S_{\text{x}} = S S_{\mathcal{T}} S'^\dagger$, $\bar{S}_{\text{x}} = \bar{S}\bar{S}_{\mathcal{T}}\bar{S}'^\dagger$
  and $\text{Tr}_{\text{w}}(S^\dagger \delta S) - \text{Tr}_{\bar{\text{w}}}(\bar{S}^\dagger \delta \bar{S}) = 0$

- For $[\mathcal{T}, P_-] \neq 0$, $[\mathcal{T}, \bar{P}_+] = 0$ where equivalence class uS, $\bar{u}_{\text{c}}\bar{S}$ with $\delta^{\text{G}}\bar{u}_{\text{c}} = 0$

$$\delta^{\text{G}}\,\mathcal{S}_{\text{eff}} = \frac{1}{2}\text{Tr}(\gamma_5\mathcal{B}) + \text{Tr}_{\text{w}}(S_{\mathcal{T}}^\dagger\, \delta^{\text{G}} S_{\mathcal{T}}) = i\,\delta^{\text{G}}\vartheta_{\mathcal{T}}$$

- No contribution by $\delta^{\text{G}} S_{\mathcal{T}}$ and $\delta^{\text{G}}\bar{S}_{\mathcal{T}}$ due to
  (1) separation of mentioned transformation
  (2) independent confirmation in the following

- Lüscher 1999, 2000 defined current $j_{\mu n}$ by

$$\mathrm{Tr}(\delta u\, u^\dagger) = -i \sum_{\mu,n} \mathrm{tr_g}(\eta_{\mu n} j_{\mu n})$$

and required it to transform gauge-covariantly

- We get explicitly $\quad \eta_{\mu n} = B^{\text{left}}_{\mu,n+\hat\mu} - U_{\mu n} B^{\text{right}}_{\mu n} U^\dagger_{\mu n}$

$$j_{\mu n} = i(U_{\mu n}\rho_{\mu n} + \rho^\dagger_{\mu n} U^\dagger_{\mu n}), \qquad \rho_{\mu n,\alpha'\alpha} = \sum_{j,\sigma} u^\dagger_{j\sigma} \frac{\partial u_{\sigma j}}{\partial U_{\mu n,\alpha\alpha'}}$$

- Requirement $j'_{\mu n} = \mathrm{e}^{B_{n+\hat\mu}} j_{\mu n} \mathrm{e}^{-B_{n+\hat\mu}}$ because of $U'_{\mu n} = \mathrm{e}^{B_{n+\hat\mu}} U_{\mu n} \mathrm{e}^{-B_n}$ implies that one must have

$$\rho'_{\mu n} = \mathrm{e}^{B_n} \rho_{\mu n} \mathrm{e}^{-B_{n+\hat\mu}}$$

which with $u' = \mathcal{T} u S_{\mathtt{x}}$ leads to the condition

$$\sum_{j,k} (S_{\mathcal{T}})^\dagger_{kj} \frac{\partial (S_{\mathcal{T}})_{jk}}{\partial U_{\mu n,\alpha\alpha'}} = 0$$

from which with $S_{\mathcal{T}}^{-1} = S_{\mathcal{T}}^\dagger$ it follows that

$$\delta^{\mathtt{G}} S_{\mathcal{T}} = 0$$

confirming previous dealing with $S_{\mathcal{T}}$ and $\bar S_{\mathcal{T}}$ from different point of view

- For $[\mathcal{T}, P_-] \neq 0$, $[\mathcal{T}, \bar P_+] = 0$ then in contrast to Lüscher $\delta^{\mathtt{G}} \mathcal{S}_{\text{eff}} = \frac{1}{2} \mathrm{Tr}(\gamma_5 \mathcal{B})$

- For perturbation expansion we get for $M=\bar{u}^\dagger D u$ with $M=M_0+M_\mathrm{I}$

$$\mathrm{det}_{\bar{\mathrm{w}}\mathrm{w}} M = \Big(1+\sum_{\ell=1}^{\infty} z_\ell\Big)\, \mathrm{det}_{\bar{\mathrm{w}}\mathrm{w}} M_0,$$

$$z_\ell = \sum_{r=1}^{\ell} \frac{(-1)^{\ell+r}}{r!} \sum_{\rho_1=1}^{\ell-r+1} \cdots \sum_{\rho_r=1}^{\ell-r+1} \delta_{\ell,\,\rho_1+\ldots+\rho_r} \frac{t_{\rho_1}}{\rho_1} \cdots \frac{t_{\rho_r}}{\rho_r}$$

$$t_\rho = \mathrm{Tr}\big((D_0^{-1}\mathcal{M})^\rho\big), \qquad \mathcal{M} = \bar{u}_0 M_\mathrm{I} u_0^\dagger$$

t_ρ fermion loops, D_0^{-1} free propagators, $\mathcal{M}$ vertices

- Vertices in more detail

  $\mathcal{M} = \bar{P}_{+0} D_\mathrm{I} P_{-0} + \bar{u}_0 \bar{u}_\mathrm{I}^\dagger D u_\mathrm{I} u_0^\dagger + \bar{u}_0 \bar{u}_\mathrm{I}^\dagger D P_{-0} + \bar{P}_{+0} D u_\mathrm{I} u_0^\dagger$

  – Inserting term $\bar{P}_{+0} D_\mathrm{I} P_{-0}$ in limit $\bar{P}_{+0}$ and P_{-0} can be replaced by $\frac{1}{2}(1+\gamma_5)$ and $\frac{1}{2}(1-\gamma_5)$

  – Other terms rely on limit of u_I and $\bar{u}_\mathrm{I}$. Since only contributions from zero and corners of the Brillouin zone survive, related chiral projections get independent of gauge field. Thus constant bases in limit and $u_\mathrm{I} \to 0$ and $\bar{u}_\mathrm{I} \to 0$

- Since only $\bar{P}_{+0} D_\mathrm{I} P_{-0}$ contributes in limit correct vertex function at tree-graph order. With appropriate Dirac operator also in general. Thus usual formulation obtained in limit