A comparative study of overlap and staggered fermions in QCD

Stephan Dürr, Christian Hoelbling, Urs Wenger

DESY Zeuthen, CPT-Marseille, NIC/DESY Zeuthen

Lattice04, Fermi National Accelerator Laboratory

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Overview

- **Introduction:**
  - Properties of the infrared fermion spectrum in QCD
  - Staggered fermions and overlap fermions - advantages and problems
  - Improvement and UV filtering
- **Infrared spectrum in lattice QCD:**
  - Dynamical ensembles
  - Quenched ensembles
  - Discussion
- **Conclusions and outlook**

Properties of the IR spectrum

- Eigenmodes of the anti-hermitian massless Dirac operator:

$$\boxed{D\psi_\lambda = i\lambda\psi_\lambda, \quad \lambda \in \boldsymbol{R}}$$

- Chiral symmetry $\{D, \gamma_5\} = 0$ and γ_5-hermiticity guarantees a spectrum symmetric around zero.

- Zero modes have definite chirality $\psi^\dagger_{\lambda=0}\gamma_5\psi_{\lambda=0} = \pm 1$.

- Index theorem relates the number of zero modes to the topological charge of the background gauge field:

$$\boxed{\frac{1}{2}\mathrm{Tr}(\gamma_5 D) = n_- - n_+ = \frac{1}{32\pi^2}\int d^4x\, \varepsilon_{\mu\nu\rho\sigma}\mathrm{Tr}F_{\mu\nu}(x)F_{\rho\sigma}(x)}$$

- Fermions are sensitive to the $U_A(1)$-anomaly via the zero modes.

- The infrared (IR) modes describe the low-energy, long-distance physics.

Staggered Fermions

- Naive discretisation of the fermionic Dirac operator produces 15 doubler fermions:

$$D^{\text{naive}}_{x,y} = \sum_\mu \gamma_\mu (U^\dagger_\mu(x)\delta_{x-\hat\mu,y} - U_\mu(x+\hat\mu)\delta_{x+\hat\mu,y})$$

- Expose the 4-fold degeneracy with the transformation $\Psi(x) \to \left(\prod_\mu \frac{1-1^{x_\mu}}{2}\gamma_\mu\right)\Psi(x)$ and remove this degeneracy by distributing the spin degrees of freedom over a hypercube:

$$D^{\text{st}}_{x,y} = \sum_\mu \eta_\mu(x) (U^\dagger_\mu(x)\delta_{x-\hat\mu,y} - U_\mu(x+\hat\mu)\delta_{x+\hat\mu,y})$$

  with $\eta_\mu(x) = (-1)^{\sum_{\mu<\nu} x_\nu}$ (Kogut, Suskind, 1975).

- 3 doubler fermions remain and we have four degenerate fermion flavours in the continuum.
- Remnant $U(1)$ chiral symmetry:

$$\chi \to e^{i\theta(-1)^{\sum x_\nu}}\chi, \qquad \bar\chi \to \bar\chi e^{-i\theta(-1)^{\sum x_\nu}}$$

Staggered Fermions

- **Advantages of staggered fermions:**
  - ✓ Remnant $U(1)$ chiral symmetry.
  - ✓ Computationally very cheap.
- **Disadvantages of staggered fermions:**
  - ✗ Remnant fermion doubling (3 doublers).
  - ✗ Doublers are light and mix.
  - ✗ No zero modes, no index theorem, no sensitivity to topology?
  - ✗ Conceptual issues:
    - Generate one flavour of sea quarks by det$(D^{\mathrm{st}})^{1/4}$?
    - How do you match this in the valence quark sector?
    - Consistent field theoretic framework or just a model of QCD?

Overlap fermions

- Circumvent the doubling problem by making the doublers heavy $\Rightarrow$ Wilson fermions, overlap fermions.
- Overlap fermions are defined by

$$\boxed{D^{\text{ov}} = 1 + \frac{D_{\text{W}}(-\rho)}{\sqrt{D_{\text{W}}^{\dagger}(-\rho) D_{\text{W}}(-\rho)}}}$$

where D_{W} is the Wilson Dirac operator (Narayanan, Neuberger 1993; Neuberger, 1997).

- Advantages of overlap fermions:

✓ Exact lattice chiral symmetry $\Rightarrow$ no mixing, no additive quark mass renormalisation.

✓ Exact zero modes $\Rightarrow$ index theorem.

✓ Correct sensitivity to topology.

- Disadvantages of overlap fermions:

✗ Computationally very expensive.

UV Filtering

- **UV filtering:** remove UV noise by smearing the gauge links

$$U_\mu(x) \to \sum_{P(x \to x+\hat{\mu})} w(P) U_P$$

(Blum et. al. 1997, MILC 1999)

- **Interpretation:**

  ⇛ Suppress the flavour changing interactions, i.e., reduce the mixing of the doubler modes.

  ⇛ Many different proposals to choose $w(P)$.

  ⇛ Smearing is an $O(a^2)$ redefinition of the fermion action if

$$\sum_P w(p) = 1 + O(a^2).$$

  ⇛ Not an improvement in the Symanzik sense.

Strategy

- **Strategy:**
  - ☞ We focus on the IR spectrum of the staggered Dirac operator.
  - ☞ Compare staggered fermions to the well defined overlap discretisation as a benchmark.
  - ☞ Find underlying pattern in the spectrum of the operator (as expected from the study in the Schwinger model).

- **Questions:**
  - ⇒ Can we see 'continuum-like' behaviour, i.e. 4-fold degeneracy, zero modes etc.?
  - ⇒ If yes, at what lattice spacing?
  - ⇒ Scaling of $O(a^2)$ artefacts?
  - ⇒ Final goal: Check if

$$\boxed{\mathrm{det}(D_{\mathrm{stag}})^{1/4} \overset{?}{\propto} \mathrm{det}(D_{\mathrm{ov}}) + O(a^2)}$$

QCD Setup

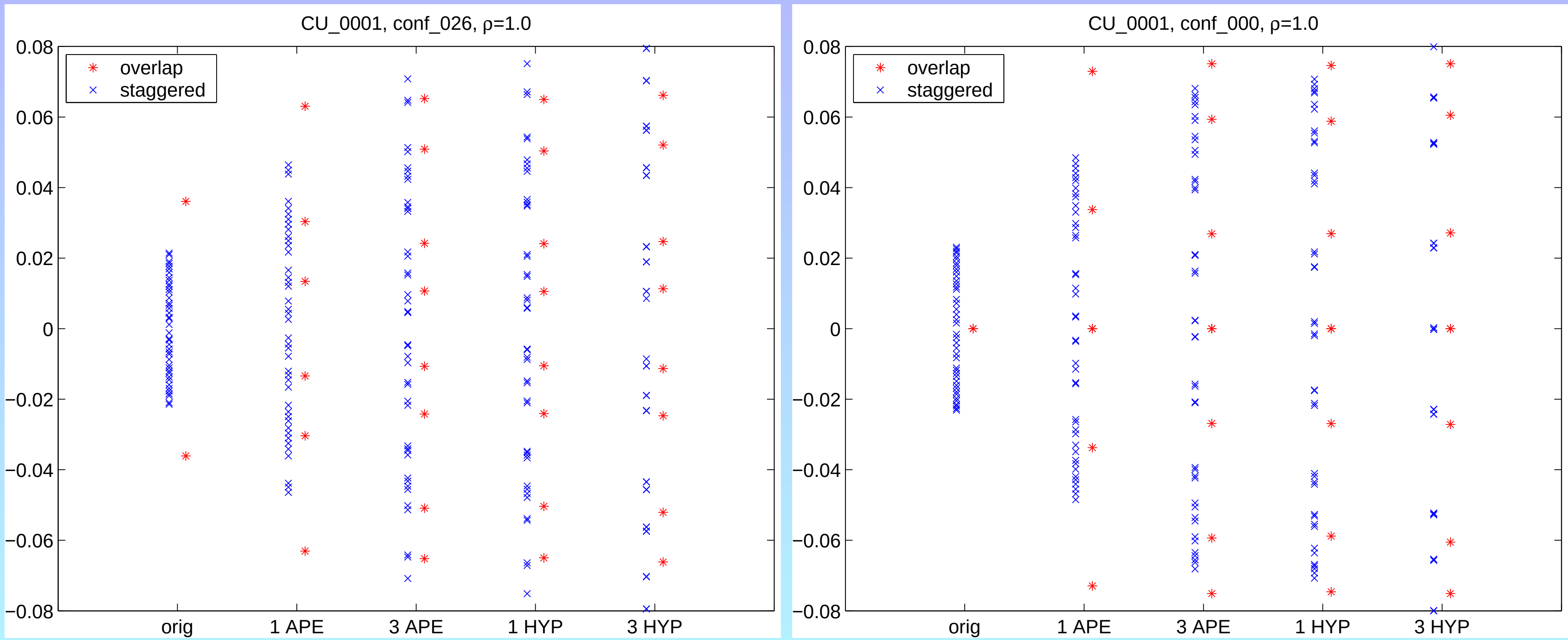
- We look at the infrared spectra of two sets of configurations:
  - ☞ $16^3 \times 32$ lattices, $\beta = 5.7$, dynamical staggered $ma = 0.01$, $a \simeq 0.1$fm from gauge connection archive
  - ☞ 4 ensembles of matched quenched lattices, $V \simeq (1.12\text{fm})^4$

β	5.66	5.79	6.0	6.18
L/a	6	8	12	16

- We use 2 standard smearing methods with originally suggested parameters
  - ☞ APE-smearing (Albanese et al. [APE Collaboration], 1987)
  - ☞ HYP-smearing (A.Hasenfratz, Knechtli, 2001)
- (We also looked at Stout smearing and non-unitarised smearing.)

IR Spectrum

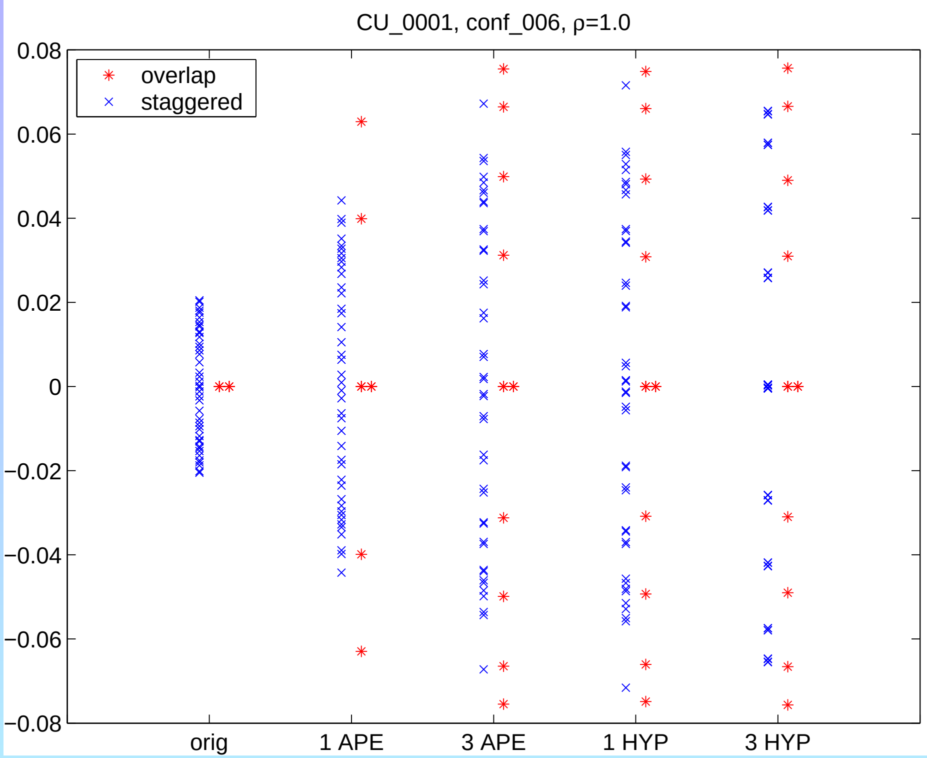
- Staggered modes (×) and overlap modes (∗) up to 160 MeV for various smearing levels:

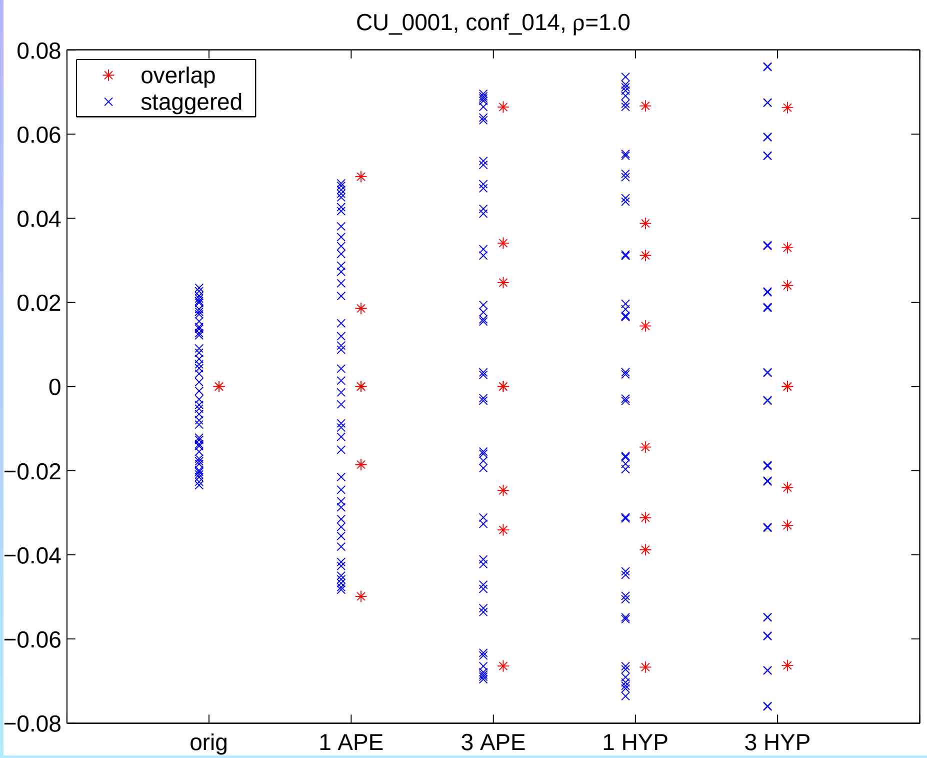


✓ Overlap spectra are very stable under smearing.

✗ No similarity in the spectra without smearing.

IR Spectrum





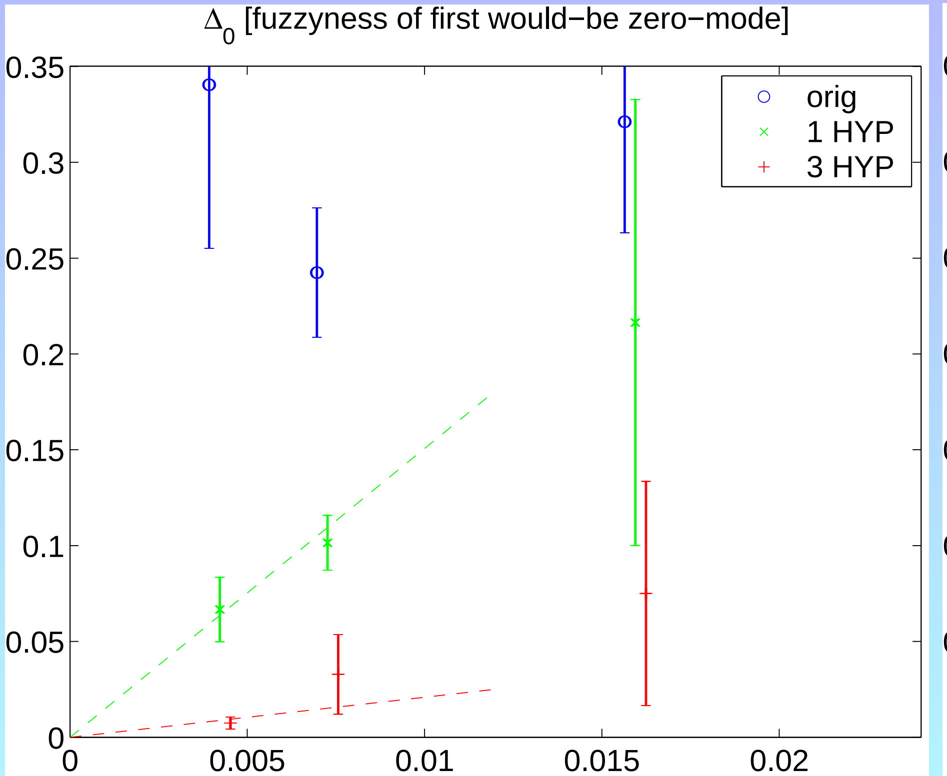
- ✓ (Approximate) 4-fold degeneracy emerges.
- ✓ Zero modes separate out $\Rightarrow$ index theorem holds for improved staggered fermions.
- ✓ Correspondence between quadruples of staggered eigenvalues and overlap eigenvalues.
- ✗ Occasional mismatch of topological charge due to dislocations $\Rightarrow O(a^2)$ artefact.

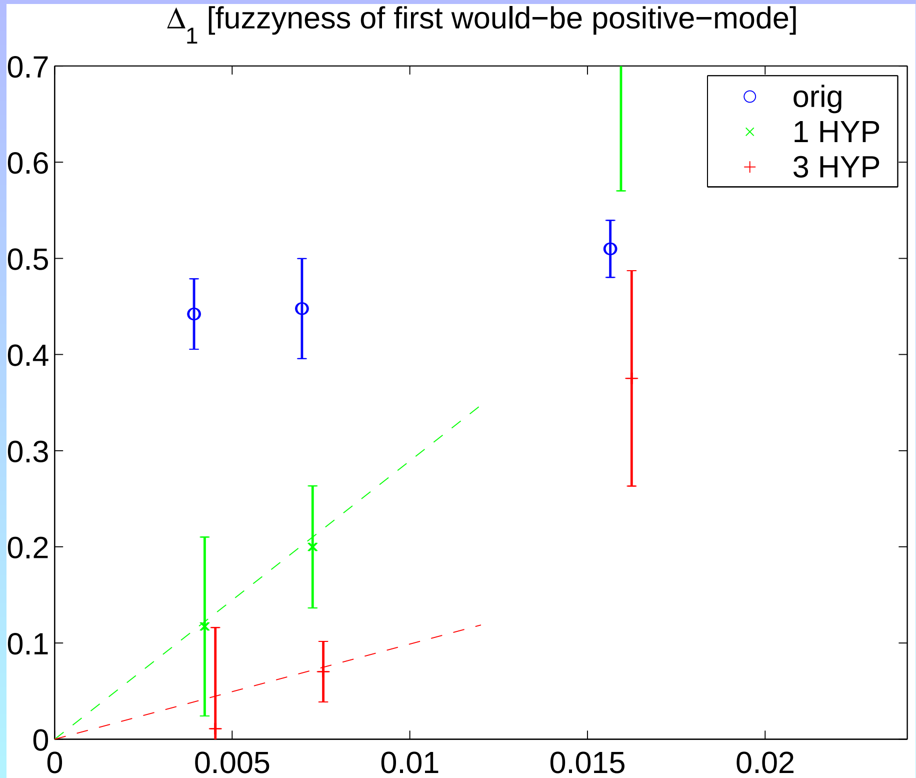
Eigenvalue Fuzziness

- Group the staggered eigenmodes into
  - Would-be zero modes $\xi_i, |\xi_{i-1}| < |\xi_i|, i = 1, \ldots, 2\nu$
  - Non-zero modes $\lambda_i, |\lambda_i| < |\lambda_{i+1}|, i = 2\nu + 1, \ldots$
- Use geometric mean to compare to the overlap eigenmodes
  - Would-be zero modes: $\sqrt{\xi_{2n-1}\xi_{2n}}$
  - Non-zero modes: $(\prod_{i=1}^4 \lambda_{4n+i-3})^{1/4}$
- We define fuzziness of a staggered eigenmode quadruple
  - Lowest would-be zero mode: $\Delta_0 = \frac{\sqrt{\xi_1\xi_2}}{(\prod_{i=1}^4 \lambda_i)^{1/4} - \sqrt{\xi_1\xi_2}}$
  - Non-zero modes: $\Delta_1 = \frac{\sqrt{\lambda_4\lambda_3} - \sqrt{\lambda_2\lambda_1}}{(\prod_{i=1}^4 \lambda_{i+4})^{1/4} - (\prod_{i=1}^4 \lambda_i)^{1/4}}$

Eigenvalue Fuzziness

- Fuzziness pseudo-observables vs. a^2:



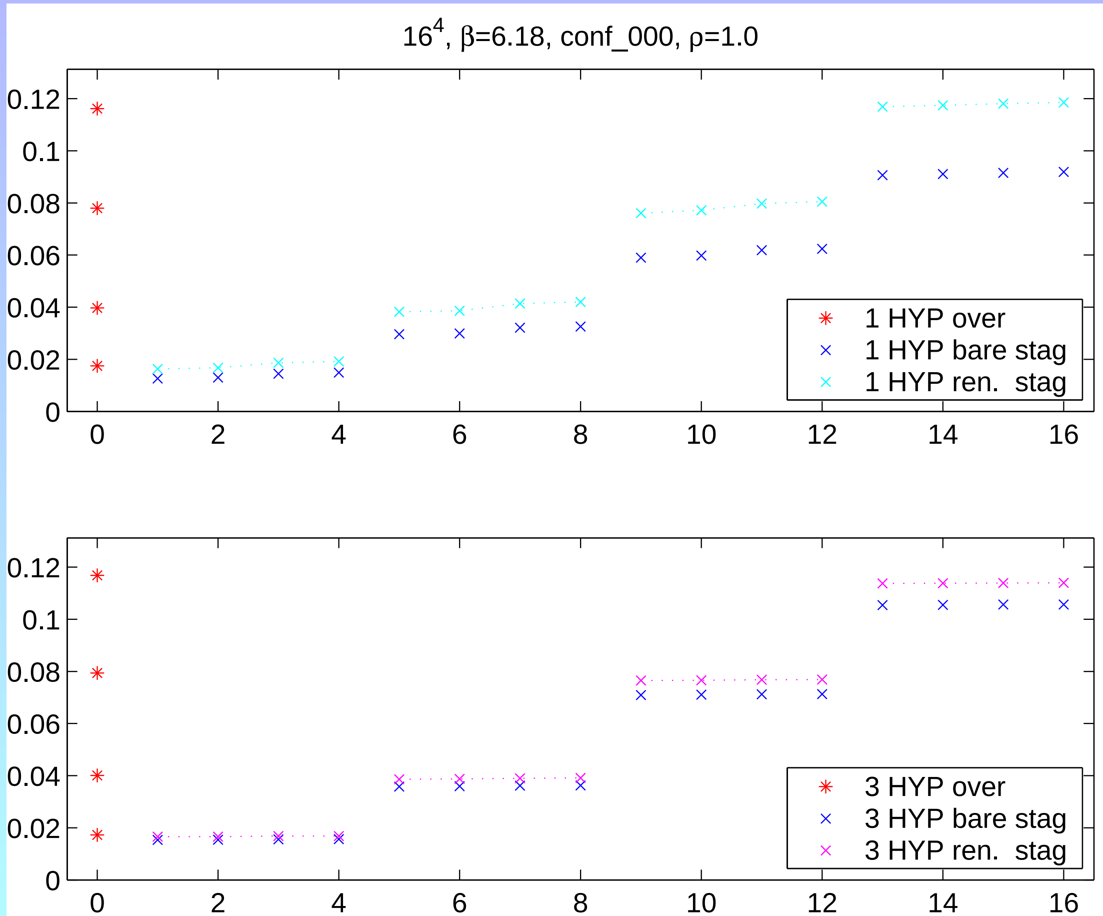


✓ Improved staggered operators: Indication for scaling.

✗ Naive staggered operator: No indication for scaling at accessible couplings.

Eigenvalue Correspondence

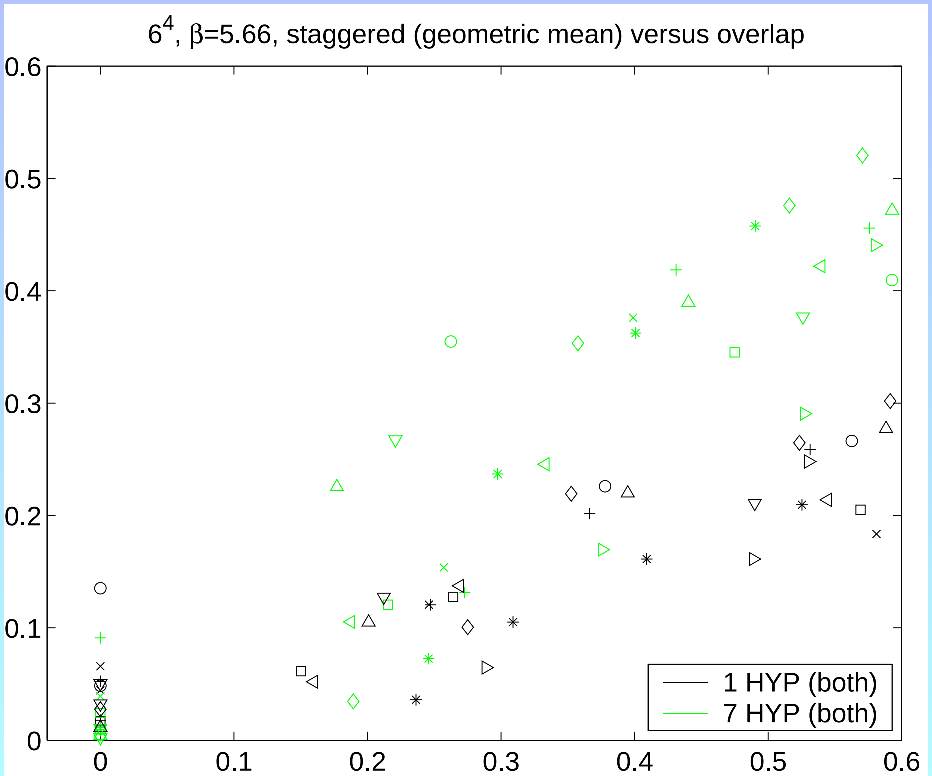
- IR spectrum of overlap and staggered Dirac operator at $\beta = 6.18$:

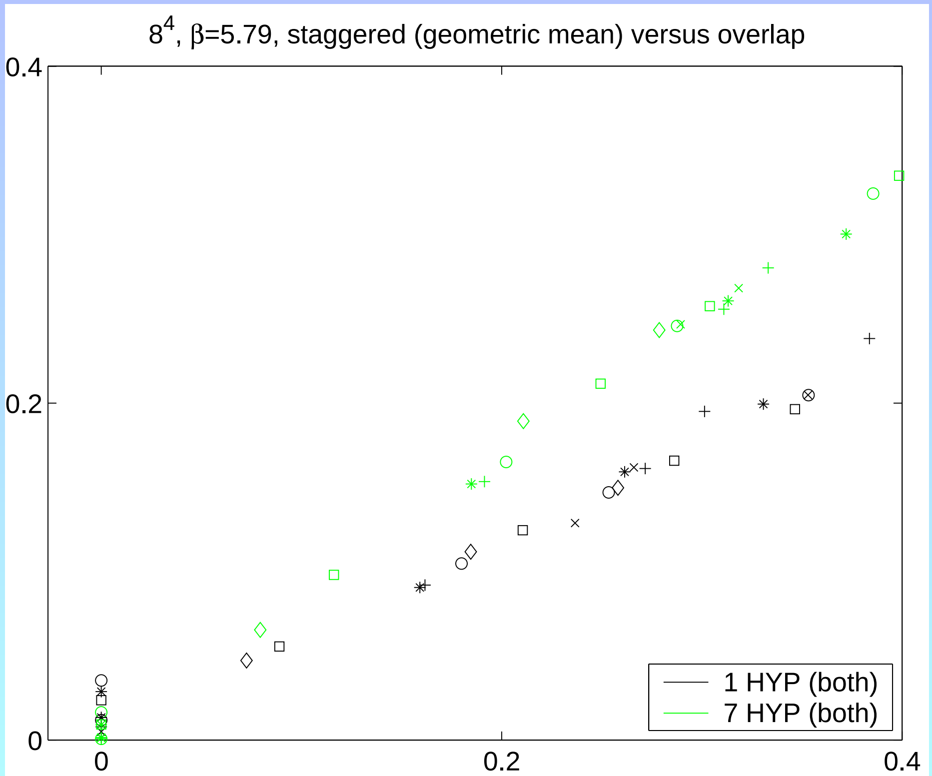


- After rescaling by a (irrelevant) renormalisation factor, all configurations on the fine lattices show a quantitative agreement between smeared overlap and staggered spectra.

Eigenvalue Correspondence

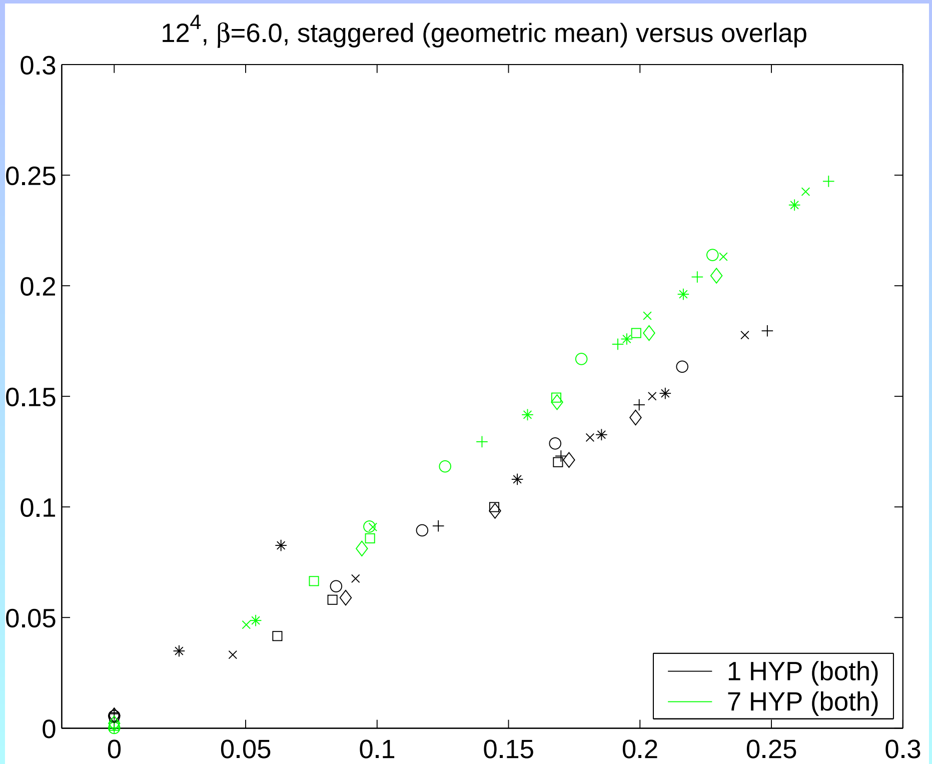
- Staggered eigenvalues vs. overlap eigenvalues (at $\beta = 5.66$ and 5.79):

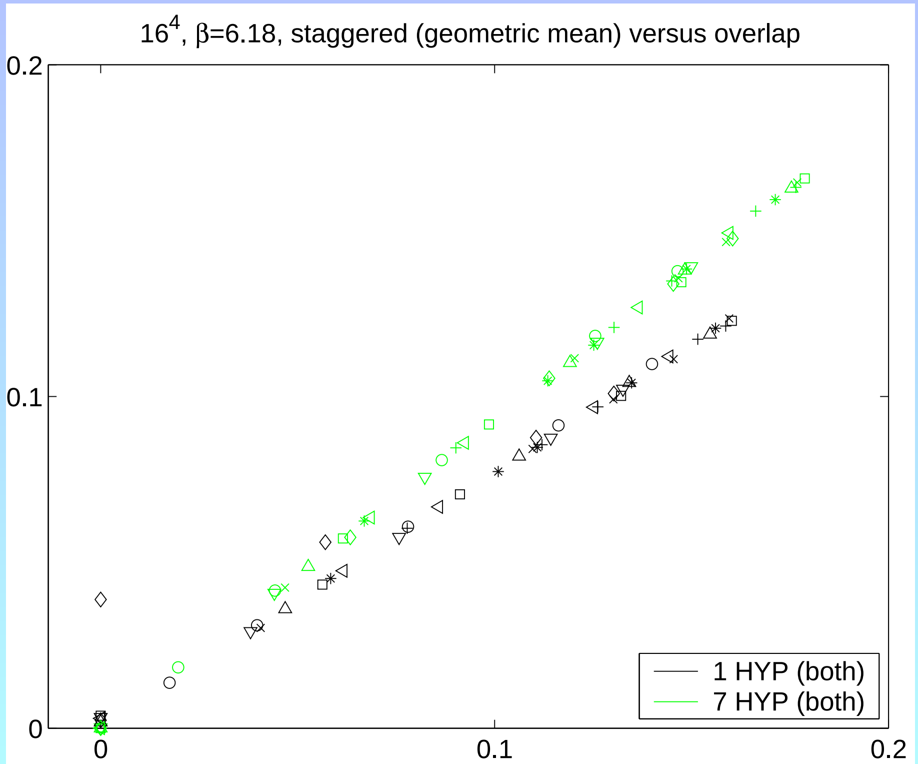




Eigenvalue Correspondence

- Staggered eigenvalues vs. overlap eigenvalues (at $\beta = 6.00$ and 6.18):





Eigenvalue Correspondence

✓ Linear correspondence between UV filtered/improved overlap eigenvalues and staggered eigenvalue quadruples

- ☞ at small, accessible couplings,
- ☞ for moderate filtering.

✓ Suggests index theorem for improved staggered fermions (up to $O(a^2)$ artefacts).

✓ Suggests correct RMT for improved staggered fermions (up to $O(a^2)$ artefacts).

✗ Prominent outliers at the origin for small smearing levels:

- ☞ Misidentification of ν by the overlap operator due to dislocations.
- ☞ This also happens between two overlap operators at different ρ
  $\Rightarrow$ typical $O(a^2)$ effect.

✗ At very large coupling, additional smearing does not improve linearity.

Conclusions and Outlook

✓ We have found evidence that the rooting-procedure of the staggered determinant is justified in the sense that

$$\boxed{\det(D_{\text{stag}})^{1/4} \propto \det(D_{\text{ov}}) + O(a^2)}$$

✓ We found a quantitative correspondence between the IR part of improved staggered and overlap spectra in QCD on a configuration by configuration basis.

✓ Our data indicates that with moderate UV filtering one is in the scaling regime at accessible couplings.

✗ Conceptual issues remain: It is not clear, whether and how a local operator D with $\det(D) = \det(D_{\text{stag}})^{1/4}$ can be constructed.

✓ **Unquench the overlap via a hybrid HMC:** use impr. staggered Dirac operator along the trajectory but accept/reject with overlap $\Rightarrow$ correct overlap ensemble is generated.