The Lattice NJL Model at Non-zero Baryon and Isospin Densities

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Abstract

We present initial results of a numerical investigation of the chiral symmetry restoring transition in the (3+1)-dimensional Nambu – Jona-Lasinio model with both non-zero baryon chemical potential ($\mu_B \neq 0$) and isospin chemical potential ($\mu_I \neq 0$). The two scales are ordered $\mu_I \ll \mu_B$, which is shown to be phenomenologically relevant in the physics of compact stars.

With non-zero isospin chemical potential, the model is thought to suffer from a sign problem. We proceed in two ways:

(i) We perform "partially quenched" simulations in which μ_I is made non-zero only during the measurement of chiral observables;

(ii) We perform full simulations with imaginary isospin chemical potential with the aim to analytically continue results to real μ_I.

Motivation

At asymptotically high baryon chemical potential (μ_B) and low temperature (T) – where QCD can be treated in a perturbative manner – the ground-state of quark matter is found to be that of a colour-superconductor (for a recent review see e.g. [1]). The determination of the ground-state at the more moderate densities typical in the cores of compact (neutron) stars would thus appear to be a perfect application for lattice QCD. Unfortunately, with $\mu_B \neq 0$ the determinant of the QCD fermion matrix $\det(\not{D} + m + \mu\gamma_0)$ becomes complex and importance sampling breaks down.

One way to proceed is to study model field theories such as the Nambu – Jona-Lasinio (NJL) model. This purely fermionic field theory, in which colour-neutral quarks interact via a four-point contact term, not only contains the same global symmetries as two flavour QCD, but can be simulated on the lattice even with $\mu_B \neq 0$. In [2] we show that the ground-state of the lattice model with $\mu_u = \mu_d = \mu_B$, i.e. with "up" and "down" quarks sharing a common Fermi surface, exhibits s-wave superfluidity via a standard BCS pairing between quarks of different flavours; i.e.

$$\langle ud \rangle \neq 0, \qquad \Delta_{BCS} \neq 0.$$

Within the cores of compact stars, however, the Fermi momenta k_F^u and k_F^d are expected to be separate. A simple argument based on that of [3] suggests that for a two flavour Fermi liquid of massless quarks with $\mu_B = 400\mathrm{MeV}$ and both weak equilibrium ($\mu_d = \mu_u + \mu_e$) and charge neutrality ($2n_u/3 - n_d/3 - n_e = 0$) enforced, all the Fermi momenta are determined:

$$\begin{aligned} k_F^u &= \mu_u = \mu_B - \mu_e/2 = 355.5\mathrm{MeV}, \\ k_F^d &= \mu_d = \mu_B + \mu_e/2 = 444.5\mathrm{MeV}, \\ k_F^e &= \mu_e = 89\mathrm{MeV}. \end{aligned}$$

The effect of separating the Fermi surfaces of pairing quarks in QCD should be to make the colour superconducting phase less energetically favourable. This could prove a good method, therefore, to investigate the stability of the superfluid phase.

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The Model

The action of the lattice NJL model (with $a \to 1$) is given by

$$\sum_{xy} \big(u(x) \; d(x) \big) \, M[\Phi, \mu_B, \mu_I]_{xy} \begin{pmatrix} u(y) \\ d(y) \end{pmatrix} + \frac{2}{g^2} \sum_{\tilde{x}} \mathrm{Tr}\Phi_{\tilde{x}}^\dagger \Phi_{\tilde{x}},$$

where (u, d) is the $SU(2)$ doublet of staggered up and down quarks defined on lattice sites x and $\Phi \equiv \sigma + i\vec{\pi}.\vec{\tau}$ is a matrix of bosonic auxiliary fields defined on dual sites $\tilde{x}$. The fermion kinetic matrix M_{xy} is defined in [2]; we use the bare parameters used therein.

One can separate the Fermi surface of up and down quarks by simultaneously setting baryon chemical potential $\mu_B \equiv (\mu_u + \mu_d)/2 \neq 0$ and isospin chemical potential $\mu_I \equiv (\mu_u - \mu_d)/2 \neq 0$. With $\mu_I = 0$, $\tau_2 M \tau_2 = M^*$, which is a sufficient, but not necessary condition to show that $\det M$ is both real and positive [4]. With $\mu_I \neq 0$ however, this is no longer true such that once again we are faced with the sign problem.

The fact that the two scales are ordered $\mu_I \ll \mu_B$ suggests that one may be able to use some of the techniques recently developed to study QCD with $\mu_B \ll T$. First, however, we present the results of a partially quenched calculation.

Partially Quenched Calculation

Whilst the primary motivation for investigating $\mu_I \neq 0$ is to study the superfluid phase, this requires one to introduce an explicit symmetry breaking term; it is currently not clear how to measure the critical temperature in the limit that this term is reduced to zero [2]. Instead, we choose to study the chiral phase transition with the aim of controlling the systematics of introducing $\mu_I \neq 0$.

The first step we take is to perform a "partially quenched" calculation in which $\mu_I = 0$ when generating the background fields and is made non-zero only during the measurement of fermion observables. In particular, we measure the up and down quark condensates

$$\langle \bar{u}u \rangle, \langle \bar{d}d \rangle \equiv \frac{1}{V} \frac{\partial \ln \mathcal{Z}}{\partial m_{u,d}} = \frac{1}{2} \left\langle \mathrm{tr}(\mathbb{1} \pm \tau_3) M^{-1} \right\rangle$$

as functions of μ_B for various μ_I on a 12^4 lattice. Some results are presented in Fig. 1.

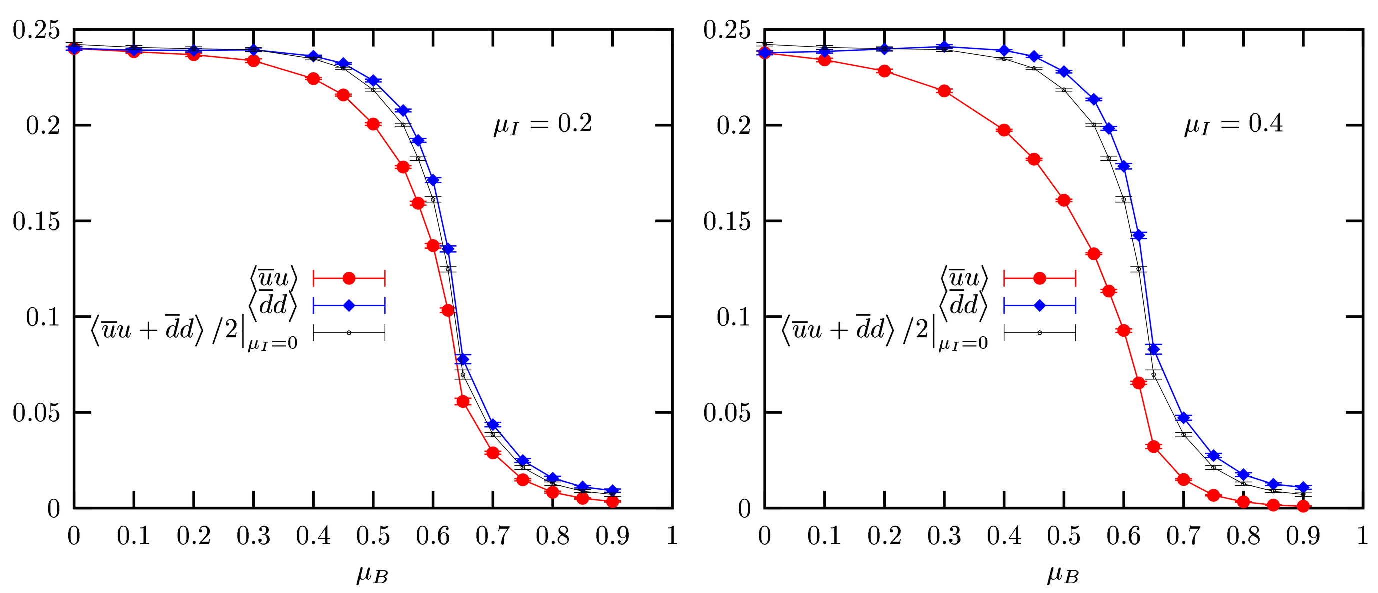

Fig. 1. $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ condensates for various μ_B and μ_I on a 12^4 lattice.

The results agree qualitatively with those of analytic studies of the model in which the introduction of a small but non-zero μ_I is seen to split the chiral phase transition into two [5, 6]. Unlike these studies, however, we find that $\langle \bar{u}u \rangle$ deviates more from the $\mu_I = 0$ result than $\langle \bar{d}d \rangle$; this may be an artifact of the partially quenched approximation.

Analytic Continuation from Imaginary μ_I

One recent method used to study QCD with small baryon chemical potential μ_B is to measure quantities at imaginary μ_B, where the measure of the path integral is real, and fit results to a truncated Taylor expansion about $\mu_B/T = 0$. One can then analytically continue the results to real μ_B [7, 8]. We propose to use a similar method by simulating at imaginary isospin chemical potential ($\tilde{\mu}_I$), where once again $\det M$ is real, and expanding observables in powers of $\tilde{\mu}_I/\mu_B$.

For our initial investigation we have measured the quark condensates and baryon and isospin number densities on a 12^4 lattice at $\mu_B = 0.6$, which from Fig. 1 can be seen to be where the effect of having $\mu_I \neq 0$ is largest. In QCD, one can show that e.g. $\langle \bar{\psi}\psi \rangle$ expanded about $\mu_B = 0$ is analytic in μ_B^2, such that for small imaginary μ_B the quantity remains real [9]. For our simulations, however, this is not the case, and measured

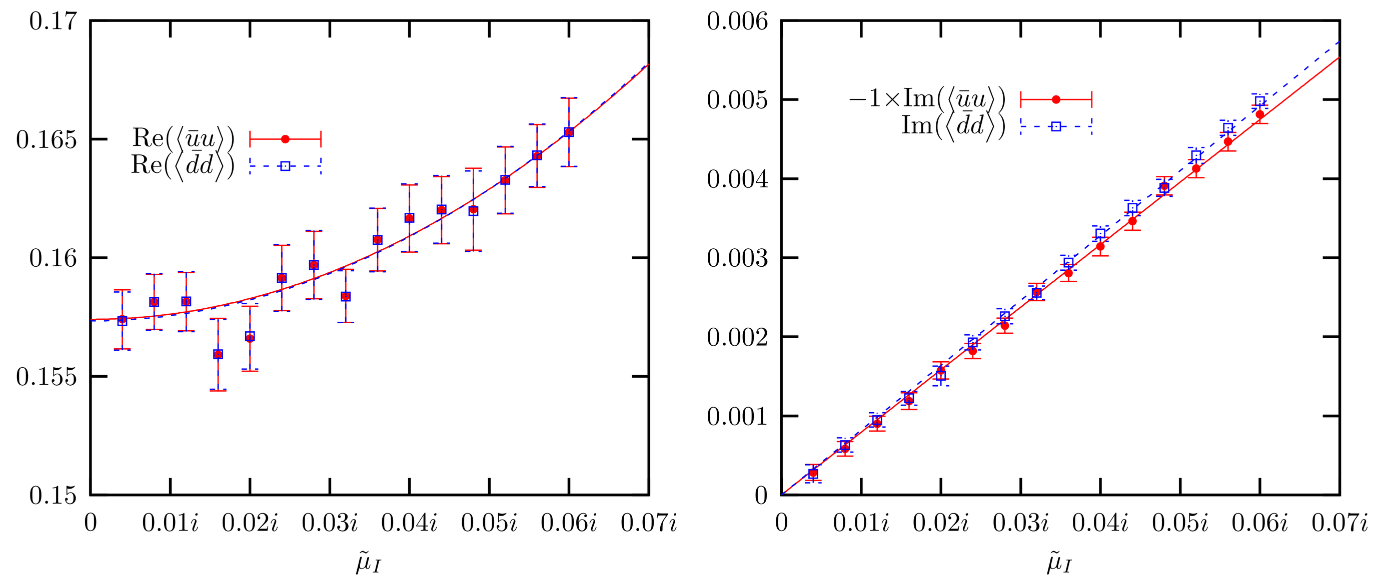

Fig. 2. Real and imaginary parts of $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ as functions of imaginary μ_I with $\mu_B = 0.6$ on a 12^4 lattice.

quantities are, in general, complex. One can therefore fit and analytically continue the results using e.g.

$$\langle \bar{u}u \rangle = \sum_{n=0}^{\infty} c_n \tilde{\mu}_I^n \;\xrightarrow{\tilde{\mu}_I \to \mu_I}\; \langle \bar{u}u \rangle|_{\mu_I \in \mathbb{R}} = c_0 + c_1\mu_I - c_2\mu_I^2 - c_3\mu_I^3 + c_4\mu_I^4 + \ldots$$

Figure 2 shows the real and imaginary parts of the quark condensates as functions of imaginary isospin chemical potential $\tilde{\mu}_I$. As expected, the results fit the forms $\mathrm{Re}\langle \bar{u}u \rangle = c_0 + c_2\tilde{\mu}_I^2$ and $\mathrm{Im}\langle \bar{u}u \rangle = c_1\tilde{\mu}_I$ well. Our next aim is, with more data and better statistics for a range of μ_B, to produce reliable forms of the curves in Fig. 1. It may also be possible to look for signs of pion condensation at larger values of μ_I.

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