Testing universality in Lattice QCD*

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Plan

1) Background

2) Conceptual/theoretical issues for staggered fermions

3) Testing universality between Wilson fermions and partially naive ($\simeq$ partially staggered) fermions

4) A specific test: comparison of lattice fermion determinants in the continuous time limit.
   $\sim$ statement of results
   $\sim$ outline of derivations
   $\sim$ comparison with known results in free field case.

5) Clarification of the "universality anomaly"

6) Something else to worry about

7) Conclusions

8) Current projects following on from this work.

Background

Major developments in lattice gauge theory during the last decade:

Conceptual–theoretical development:

The long-standing problem of how to formulate chiral symm. on lattice has been solved. Explicit solution: Overlap fermions – have exact chiral symm. encapsulated in GW relation $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$
$\longrightarrow$ free of the problems caused by lack of chiral symm. in previous formulations.
However, implementing Overlap fermions in computer simulations is v. difficult challenge. Further advances in algorithms & computer power needed...

Practical–theoretical development:

Algorithm advances for improved staggered fermions have made possible dynamical fermion simulations on realistically large lattices. Impressive increase in agreement with experimental values for hadron masses compared to previous quenched simul's.
However, problematic conceptual/theoretical issues...
$\longrightarrow$ Should the results be regarded as coming from first-principles calculation in QCD, or instead from some kind of effective theory?

Theoretical issues for staggered fermions

1) Staggered formulation is lattice theory for 4 continuum fermion flavours.
$\rightarrow$ need to take fractional power of fermion determinant to describe 2 or 3 flavours of dynamical (sea) quarks.
Does this mean we are dealing with a lattice theory with non-local action?

2) Is perturbative lattice QCD with staggered fermions renormalisable? And if so, does it reproduce 4 flavour continuum QCD?

– This is well-established for perturbative lattice QCD with Wilson fermions [Reisz,'86-'88]. But the staggered fermion case remains unresolved – has extra complications:

- $\sim$ The lattice quark flavour fields live on the blocked lattice (spacing $= 2a$), while the gauge link variables live on the original lattice (spacing $= a$).
- $\sim$ The lattice action contains flavour-changing interactions. Although these terms vanish in the naive continuum limit, it is far from clear that the Feynman diagrams involving them will all vanish in the continuum limit of a renormalised theory.

Testing universality with partially naive fermions

Euclidean spacetime lattice, finite volume.
a=time lattice spacing, a'=spacial lattice spacing

$$
\begin{aligned}
S_{fermion} &= a(a')^3 \sum_{(\mathbf{x},\tau)} \bar{\psi}(\mathbf{x},\tau) D^{(r)} \psi(\mathbf{x},\tau) \\
D^{(r)} &= \tfrac{1}{a}\gamma_4 \nabla_4 + \tfrac{r}{2a}\Delta_4 + D_{space} + m \\
D_{space} &= \tfrac{1}{a'}\nabla_{space} + \tfrac{r'}{2a'}\Delta_{space} \\
\nabla_\mu \psi(x) &= \tfrac{1}{2}[\,U_\mu(x)\psi(x+b\hat{\mu}) - U_\mu(x-b\hat{\mu})\psi(x-b\hat{\mu})] \\
\Delta_\mu \psi(x) &= 2\psi(x) - U_\mu(x)\psi(x+b\hat{\mu}) - U_\mu(x-b\hat{\mu})\psi(x-b\hat{\mu})
\end{aligned}
$$

$$b \in \{a, a'\}.$$

$r = r' \neq 0 \;\Rightarrow\;$ Wilson fermions
$r = r' = 0 \;\Rightarrow\;$ naive lattice fermions
$r=0$, $r' \neq 0 \Rightarrow$ partially naive–partially Wilson fermions

$r = 0 \rightarrow$ "fermion doubling" on Euclidean time axis.

$\rightarrow$ Can test universality by comparing the $r=0$ lattice theory with the theory for 2 flavours of Wilson ($r \neq 0$) fermions: According to universality these should coincide in the continuous time ($a \rightarrow 0$) limit.

Partially staggered interpretation in $r=0$ case

Lattice Dirac operator

$$D^{(r)} \;=\; \frac{1}{a}\gamma_4\nabla_4 + \frac{r}{2a}\Delta_4 + D_{space} + m$$

In $r=0$ case introduce staggered 'flavour' fields:

$$\psi_1(\mathbf{x},\tau) := \psi(\mathbf{x},2\tau) \quad , \quad \psi_2(\mathbf{x},\tau) := \psi(\mathbf{x},2\tau+a)$$

living on partially blocked lattice. (time spacing$=2a$)
In free field case $D^{(r=0)}$ then takes form

$$D^{(r=0)}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \left[\frac{1}{a}\gamma_4\begin{pmatrix}0 & \partial_4^-\\ \partial_4^+ & 0\end{pmatrix} + D_{space} + m\right]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Now make basis transform in spinor$\otimes$flavour space:

$$D^{(r=0)} \;\to\; \mathcal{O}^{-1}D^{(r=0)}\mathcal{O} \qquad\qquad \mathcal{O} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\gamma_5\gamma_4 & 0\\ 0 & 1\end{pmatrix}$$

$$= \gamma_4\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{1}{2a}\partial_4 + \gamma_5\otimes\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{1}{2(2a)}\Delta_4 + D_{space} + m$$

where $\partial_4 = \frac{1}{2}(\partial_4^+ + \partial_4^-)$. This is 'partially staggered' lattice Dirac operator. $\to$ Any problems with usual staggered fermions are likely to also show up in this setting.

Testing universality by comparing lattice fermion det's

Universality implies

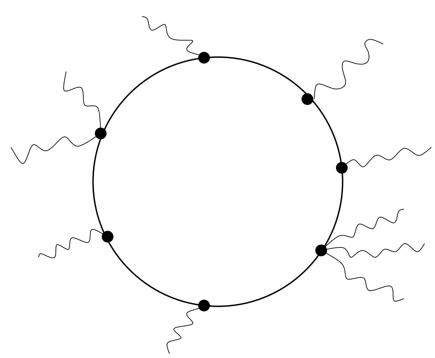
$$\lim_{a\to 0}\ detD^{(r=0)} = \Big(\lim_{a\to 0}\ detD^{(r\neq 0)}\Big)^2 \qquad \text{mod p.i.f.'s}$$

– Will check this by explicit analytic calculation.

Although the fermion det is not a physical observable, there are several reasons why this test is of interest:

1) It tests the "fractional power" prescription for the fermion det in lattice QCD with staggered fermions.

2) The fermion det contains the one (fermion) loop gluonic n-point functions:



$\rightarrow$ Problems for renormalisation of LQCD with (partially) staggered fermions could manifest themselves here.

Universality expectation:

$$\lim_{a\to 0}\ detD^{(0)} = \Big(\lim_{a\to 0}\ detD^{(r\neq 0)}\Big)^2 \qquad \text{mod p.i.f.'s}$$

<u>Result of direct calculation</u> (mod p.i.f.'s):

$$\lim_{a\to 0}\ detD^{(0)} = \Big(\lim_{a\to 0}\ detD^{(\pm 1)}\Big)^2\, e^{\mp\int_0^\beta Tr\,(\frac{r'}{2a'}\Delta_{space}(\tau))\,d\tau}$$

$\longrightarrow$ reveals "universality anomaly". Phys. significance?

$\Delta_{space}(\tau)$ defined on QM Hilbert space $\{\psi(\mathbf{x})\}$.
(replace $\psi(\mathbf{x},\tau)$ by $\psi(\mathbf{x})$ in def. of $\Delta_{space}\psi(\mathbf{x},\tau)$)
– involves spacial link variables $U_\sigma(\mathbf{x},\tau)$ $(\sigma=1,2,3)$.

$\frac{r'}{2a'}\Delta_{space}$ formally vanishes in spacial continuum limit $a'\to 0$, but in actual fact $Tr\frac{1}{a'}\Delta_{space}$ diverges in this limit since the largest eigenvalue of $\frac{1}{a'}\Delta_{space}$ is $\sim\frac{1}{a'}$.

Will come back later to the physical significance, or lack thereof, of this anomaly factor.

The results in more detail

β := length of time axis; held fixed when taking the continuous time limit $a \to 0$. $(\beta = aN_\beta)$.
$N :=$ dim. of QM Hilbert space.
$U_4(\mathbf{x},\tau) =$ lattice transcript of $A_4(\mathbf{x},\tau)$.

By direct calculation we find:

$$\boxed{\begin{aligned} det D^{(r=\pm 1)} &\xrightarrow{a\to 0} \left(\tfrac{1}{a}\right)^{NN_\beta} e^{\pm\int_0^\beta TrM(\tau)\,\mathrm{d}\tau}\, det(\mathbf{1}-\mathcal{V}(\beta)) \\ det D^{(r=0)} &\xrightarrow{a\to 0} \left(\tfrac{1}{2a}\right)^{NN_\beta} det(\mathbf{1}-\mathcal{V}(\beta))^2 \end{aligned}}$$

$M(\tau) = \frac{r'}{2a'}\Delta_{space}(\tau) + m$, and $\mathcal{V}(\tau)$ for each $\tau \in \mathbf{R}$ is an operator on QM Hilbert space $\{\psi(\mathbf{x})\}$ given as follows:
Regard $\psi(\mathbf{x},\tau)$ as function $\Psi(\tau)$ taking values in QM Hilbert space. Introduce continuous time—lattice space Dirac operator:

$$D = \gamma_4\left(\frac{\partial}{\partial\tau} + A_4(\tau)\right) + D_{space}(\tau) + m \qquad \tau \in [0,\beta]$$

Extend to all $\tau \in \mathbf{R}$ by periodicity under $\tau \to \tau + \beta$. Then:

$$D\Psi(\tau) = 0 \quad (\tau \in \mathbf{R}, \text{ no periodicity requirement on } \Psi(\tau))$$

$$\Leftrightarrow \quad \Psi(\tau) = \mathcal{V}(\tau)\Psi(0) \qquad \leftarrow \text{definition of } \mathcal{V}(\tau)$$

Outline of the derivation: step 1

$\Psi(\tau)$ ($\leftrightarrow \psi(\mathbf{x},\tau)$) and $D^{(r)}$ represented as

$$\hat{\Psi} = (\hat{\Psi}(0), \dots, \hat{\Psi}(N_\beta - 1)) \quad , \qquad \hat{\Psi}(k) = \Psi(ak)$$

$$\hat{D}^{(r)}\hat{\Psi}(k) = d^{(r)}_{-1}(k)\hat{\Psi}(k-1) + d^{(r)}_0(k)\hat{\Psi}(k) + d^{(r)}_1(k)\hat{\Psi}(k+1)$$

Write $\hat{D}^{(r)}$ as $N_\beta \times N_\beta$ matrix and evaluate determinant by method of P. Gibbs [Phys. Lett. B172 (1986) 53].

Note: $d^{(r)}_{\pm 1}(k) \sim (\gamma_4 \mp r)$ invertible for $r \neq \pm 1$ but not for $r = \pm 1$.
$\rightarrow$ The cases $r = \pm 1$ and $r \neq \pm 1$ require separate treatments. Result is (with N_β even in $r \neq \pm 1$ case):

$$det D^{(r\neq\pm 1)} = \left(\frac{(1-r^2)^2}{2a}\right)^{NN_\beta} det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \hat{\mathcal{U}}^{(r)}(N_\beta/2)\right)$$

$$det D^{(r=\pm 1)} = \left(\frac{1}{a}\right)^{NN_\beta} \chi_\pm(M)\, det(\mathbf{1} - \hat{\mathcal{V}}(N_\beta))$$

where $\chi_\pm(M) := \prod_{k=0}^{N_\beta - 1} det(\mathbf{1} \pm a\hat{M}(k))^{1/2}$.
$\hat{\mathcal{U}}^{(r)}$ is evolution op. for $\hat{D}^{(r\neq\pm 1)}\hat{\Psi}(k) = 0 \quad (k \in \mathbf{Z})$:

$$\begin{pmatrix} \hat{\Psi}(2n) \\ \hat{\Psi}(2n+1) \end{pmatrix} = \hat{\mathcal{U}}^{(r)}(n) \begin{pmatrix} \hat{\Psi}(0) \\ \hat{\Psi}(1) \end{pmatrix}$$

(two initial values needed)

$\hat{\mathcal{V}}(k)$ evolution op. for $\hat{D}^{(\pm 1)}\hat{\Psi}(k) = 0 \quad (k \in \mathbf{Z})$:

$$\hat{\Psi}(k) = \hat{\mathcal{V}}(k)\hat{\Psi}(0)$$

(one initial value needed)

Outline of the derivation: step 2

Show that the lattice time evol. op's $\hat{\mathcal{U}}^{(r)}(N_\beta/2)$ and $\hat{\mathcal{V}}(N_\beta)$ converge to appropriate continuous time evol. op's. Can do this if $\hat{D}^{(r)}$ can be written in the form

$$\hat{D}^{(r)} = \hat{L}_1(k)\,\tfrac{1}{a}\,\partial + \hat{L}_0(k) \qquad\qquad (*)$$

with ∂ = forward or backward finite diff. op., and $\hat{L}_j(k)$ ($k \in \mathbf{Z}$, $j = 1,2$), which act on the QM Hilbert space, are such that

$$\hat{L}_j(k) = L_j(ak) + O(a)$$

for some $L_j(\tau)$ ($\tau \in \mathbf{R}$). Then the solutions to $\hat{D}^{(r)}\hat{\Psi}(k) = 0$ approximate the solutions to $D\Psi(\tau) = 0$ where

$$D = L_1(\tau)\,\tfrac{d}{d\tau} + L_0(\tau)$$

Consequently the lattice time evol. op. converges to the corresponding continuous time evol. op. for $a \to 0$.

This type of convergence result known in math lit.: Burghelea et.al. [CMP 138 (1991) 1]; Forman [CMP 147 (1992) 485].

Have only been able to write $\hat{D}^{(r)}$ in form $(*)$ in the $r = 0$ and $r = \pm 1$ cases.

Outline of the derivation: step 2 (cont'd)

$r = 1$ case $\quad$ ($r = -1$ similar). $\hat{\Psi} = \begin{pmatrix} \hat{\Psi}_+ \\ \hat{\Psi}_- \end{pmatrix}$, $\gamma_4 \hat{\Psi}_\pm = \pm \hat{\Psi}_\pm$

$$\hat{D}^{(r=1)} = \hat{L}_1 \frac{1}{a} \begin{pmatrix} \partial^- & 0 \\ 0 & \partial^+ \end{pmatrix} + \hat{L}_0$$

$$\hat{L}_1(k) = \gamma_4 \begin{pmatrix} \hat{U}_4(k-1)^{-1} & 0 \\ 0 & \hat{U}_4(k) \end{pmatrix} \qquad (\hat{U}_4(k)\psi(\mathbf{x}) := U_4(\mathbf{x},ak)\psi(\mathbf{x}))$$

$$\hat{L}_0(k) = \gamma_4 \begin{pmatrix} \frac{1}{a}(1-\hat{U}_4(k-1)^{-1}) & 0 \\ 0 & \frac{1}{a}(\hat{U}_4(k)-1) \end{pmatrix} + \hat{D}_{space}(k) + m$$

$$\longrightarrow \hat{L}_j(k) = L_j(ak) + O(a)\,,$$

$$D = L_1(\tau)\frac{d}{d\tau} + L_0(\tau) = \gamma_4(\frac{d}{d\tau} + A_4(\tau)) + D_{space}(\tau) + m$$

$$\Rightarrow \lim_{a\to 0} \hat{\mathcal{V}}(N_\beta) = \mathcal{V}(\beta)$$

$r = 0$ case:

$$\hat{D}^{(r=0)} = \hat{L}_1 \frac{1}{2a} \begin{pmatrix} \partial^+ & 0 \\ 0 & \partial^- \end{pmatrix} + \hat{L}_0 \quad \text{on} \begin{pmatrix} \hat{\Psi}_1(n) \\ \hat{\Psi}_2(n) \end{pmatrix} = \begin{pmatrix} \hat{\Psi}(2n) \\ \hat{\Psi}(2n+1) \end{pmatrix}$$

$$\hat{L}_1(n) = \gamma_4 \begin{pmatrix} 0 & \gamma_4 \hat{U}_4(2n-1)^{-1} \\ \gamma_4 \hat{U}_4(2n+1) & 0 \end{pmatrix}$$

$$\hat{L}_0(n) = \begin{pmatrix} \hat{D}_{space}(2n)+m & \gamma_4 \frac{1}{2a}(\hat{U}_4(2n+1) - \hat{U}_4(2n)^{-1}) \\ \gamma_4 \frac{1}{2a}(\hat{U}_4(2n) - \hat{U}_4(2n-1)^{-1}) & \hat{D}_{space}(2n)+m \end{pmatrix}$$

$$\longrightarrow \hat{L}_j(k) = L_j(2ak) + O(a)\,,$$

$$\widetilde{D} = \begin{pmatrix} D_{space}(\tau)+m & \gamma_4(\frac{d}{d\tau}+A_4(\tau)) \\ \gamma_4(\frac{d}{d\tau}+A_4(\tau)) & D_{space}(\tau)+m \end{pmatrix} = \mathcal{O}^{-1} \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \mathcal{O}$$

$$\Rightarrow \lim_{a\to 0} \hat{\mathcal{U}}^{(r=0)}(N_\beta/2) = \mathcal{O} \begin{pmatrix} \mathcal{V}(\beta) & 0 \\ 0 & \mathcal{V}(\beta) \end{pmatrix} \mathcal{O}^{-1}$$

Free field case

$\mathcal{V}(\beta) = e^{-\beta H}$, $\quad H = \gamma_4(\nabla_{space} + M)$ QM Hamiltonian

energy eigenvalues $\quad \pm E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + M_{\mathbf{p}}}$ $\qquad$ (2-fold degen.)

$$\begin{aligned} det(1 - \mathcal{V}(\beta)) &= \prod_{\mathbf{p}} \left[(1 + e^{\beta E_{\mathbf{p}}})(1 + e^{-\beta E_{\mathbf{p}}}) \right]^2 \\ &= \prod_{\mathbf{p}} \left[e^{\beta E_{\mathbf{p}}} (1 + e^{-\beta E_{\mathbf{p}}})^2 \right]^2 \\ &= det(D)_{Matsubara} \end{aligned}$$

This is precisely the expression for free field det. which arises from Matsubara frequency summation method (after discarding the undetermined overall constant which appears in that approach).

Hence, in free field case,

$$detD^{(r=\pm 1)} \quad \xrightarrow{a\to 0} \quad \left(\tfrac{1}{a}\right)^{NN_\beta} e^{\pm \int_0^\beta TrM(\tau)\,\mathrm{d}\tau}\, det(D)_{Matsub.}$$

$$detD^{(r=0)} \quad \xrightarrow{a\to 0} \quad \left(\tfrac{1}{2a}\right)^{NN_\beta} det(D)^2_{Matsub.}$$

This can also be derived starting from $detD^{(r)} = product\ of\ eigenvalues$.
$\longrightarrow$ provides crosscheck on correctness of the results.

Clarification of the "universality anomaly"

$$\lim_{a\to 0} \, detD^{(0)} = \Big(\lim_{a\to 0} \, detD^{(\pm 1)} \Big)^2 \, e^{\mp \int_0^\beta Tr\,(\frac{r'}{2a'}\Delta_{space}(\tau))\, d\tau}$$

"Anomaly" factor appears to depend on the gauge field. If this is really the case then serious problem for universality (seen, e.g., at perturbative level by expanding the relation in powers of the gauge field).

However, although Δ_{space} depends on gauge field, it turns out that $Tr\,\Delta_{space}(\tau)$ is independent of the gauge field! The anomaly factor is therefore physically inconsequential.

Proof:

$$\begin{aligned}
\Delta_{space}(\tau) &= \textstyle\sum_{\sigma=1,2,3} \Delta_\sigma(\tau) \\
\Delta_\sigma(\tau)\psi(\mathbf{x}) &= 2\psi(\mathbf{x}) - U_\sigma(\mathbf{x},\tau)\psi(\mathbf{x}+a'\hat{\sigma}) \\
&\quad - U_\sigma(\mathbf{x}-a'\hat{\sigma})^{-1}\psi(\mathbf{x}-a'\hat{\sigma}) \\
Tr\,\Delta_\sigma(\tau) &= \textstyle\sum_{\mathbf{p}} \langle \psi_{\mathbf{p}}\,, \Delta_\sigma \psi_{\mathbf{p}} \rangle \qquad \psi_{\mathbf{p}}(\mathbf{x}) \sim e^{i\mathbf{p}\mathbf{x}} \\
&= \textstyle\sum_{\mathbf{p}} \sum_{\mathbf{x}} \big[2 - U_\sigma(\mathbf{x},\tau)\, e^{ip_\sigma a'} \\
&\qquad - U_\sigma(\mathbf{x}-a'\hat{\sigma})^{-1}\, e^{-ip_\sigma a'} \big]
\end{aligned}$$

The terms involving U_σ vanish since $\sum_{p_\sigma} e^{ip_\sigma a'} = \delta(a') = 0$!

Something else to worry about:

In the derivation of the $a \to 0$ limit of $detD^{(r=0)}$ it was assumed that N_β is even. Recall:

$$detD^{(r=0)} = \left(\tfrac{1}{2a}\right)^{NN_\beta} det\left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} - \hat{\mathcal{U}}^{(r)}(N_\beta/2)\right]$$

$$detD^{(r=0)} \xrightarrow{a\to 0} \left(\tfrac{1}{2a}\right)^{NN_\beta} det(1-\mathcal{V}(\beta))^2 \qquad (N_\beta \text{ even})$$

One would expect that the continuous time limit with odd N_β is the same.
However, this turns out not to be the case!
For odd N_β one finds

$$detD^{(r=0)} = \left(\tfrac{1}{2a}\right)^{NN_\beta} det\left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & 1 \\ 1 & \mathbf{0} \end{pmatrix} \hat{\mathcal{U}}^{(r)}(N_\beta/2)\right]$$

and subsequently

$$detD^{(r=0)} \xrightarrow{a\to 0} \left(\tfrac{1}{2a}\right)^{NN_\beta} det(1-\mathcal{V}(\beta))\, det(1+\mathcal{V}(\beta))$$

$\longrightarrow$ Continuous time limit of partially staggered fermion determinant $detD^{(r=0)}$ is ambiguous: different answers are obtained depending on whether N_β is restricted to be even or odd!

Conclusions

A partially staggered lattice formulation for 2 fermion flavours was considered. Universality of LQCD can be tested by comparing this with the lattice theory for 2 flavours of Wilson fermions in the continuous time limit.

Universality relation for fermion determinants,

$$\lim_{a\to 0}\ detD^{(0)} = \Big(\lim_{a\to 0}\ detD^{(\pm 1)}\Big)^2 ,$$

was found to hold mod p.i.f.'s.

Implications:
– Lends support for the "fractional power" prescription for fermion det in LQCD with staggered fermions.
– Gives a first indication that a perturbatively renormalisable theory for (at least partially) staggered fermions exists and is in the right universality class.

The relevance goes beyond the specific issues for staggered fermions – tests general hypothesis that lattice theories whose actions and propagators have same naive continuum limit are in the same universality class.

Current projects following on from this work:

– Perturbative renormalisation of lattice QCD with partially staggered fermions (i.e. Dirac operator $D^{(r=0)}$). Aim to show that this is in same universality class as renormalised theory with 2 flavours of Wilson fermions. (This would be a first step toward renormalisation of usual staggered fermion theory.)

Renormalisation of a variant of partially staggered fermions in the context of chiral sigma model (an effective field theory; no gauge fields involved) has been done previously by Pernici et.al. [NPB 458 (1996) 355, hep-th/9507055].

–The specific formulae derived for $\lim_{a\to 0} detD^{(r)}$, involving the evolution operator $\mathcal{V}(\beta)$, can be used to show equivalence between Hamiltonian and functional integral formulations of lattice QCD. (Fills gaps in previous treatments.)